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Time Averaging as an Approximate Technique for Constructing Quasi-Gasdynamic and Quasi-Hydrodynamic Equations

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Abstract—Quasi-gasdynamic and quasi-hydrodynamic equations for compressible gas flows and viscous incompressible fluid flows are constructed by averaging the corresponding Navier–Stokes equations over time with the use of certain approximations.

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INTRODUCTION

In [1–6] two interrelated systems of equations were proposed and studied, namely, the quasi-gasdynamic (QGD) and quasi-hydrodynamic (QHD) equations. These systems have proved highly effective for the numerical simulation of compressible and viscous incompressible flows.

A number of techniques for constructing the QGD and QHD systems were presented in [1–6] (see also the references therein). The QGD equations were first constructed in 1985 on the basis of a kinetic model involving free-molecular motion and Maxwellization. Later, they were derived using a difference scheme for the Boltzmann equation, a regularized Boltzmann equation in the BGK form, and conservation laws written for a small stationary volume of gas. However, all these methods for constructing the QGD equations are not rigorous. In contrast, the QHD equations were rigorously derived by Yu.V. Sheretov in 1994 as based on the classical postulates of fluid dynamics (see [3, 6]). Later, he constructed these equations approximately with the use of conservation laws for a small stationary volume. The rigorous derivation of the QHD equations implies that the Navier–Stokes equations can be obtained as a consequence of the QHD equations.

An analysis shows that QGD and QHD systems and the Navier–Stokes equations are closely interrelated and do not contradict each other. Specifically, the correspondence between certain exact solutions of the QGD and QHD systems and the Navier–Stokes and Euler equations was analyzed in [3, 6]. In [7, 8] exact solutions of the QHD equations were constructed that satisfy the Navier–Stokes and Euler equations.

In this paper, we propose a new approximate method for constructing QGD and QHD equations. The method is based on the averaging (or smoothing) of the classical Navier–Stokes equations over a short time interval and has a number of advantages over available unrigorous approaches. Specifically, with this method, all previously constructed versions of the QGD and QHD equations can be derived in a rather simple unified manner. Moreover, the method explains the nature of the arising regularizing additions and can be used to expand the family of QGD and QHD equations.

Note that the various time-averaging techniques are a well-known method for transforming equations in mechanics. Specifically, time averaging is used to derive the Reynolds equations for turbulent flows (see [9]).

Below, the averaging approach is described in detail and the corresponding assumptions are formulated. First, smoothed equations are constructed in the simplest case of viscous incompressible flows. Then we sequentially construct the QHD and QGD equations for compressible gas flows. Mathematically, the approach is somewhat similar to the transformations performed for the first time in the finite volume method (see [3, 6]) and later developed in [4, 5], since the time averaging of gasdynamic fluxes was implicitly present in this method.

1. QHD EQUATIONS FOR VISCOUS INCOMPRESSIBLE FLOWS IN THE OBERBECK–BOUSSINESQ APPROXIMATION

The viscous incompressible Navier–Stokes equations in the Oberbeck–Boussinesq approximation have the form (see [9, 10])

$$\mathrm{div}\mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathrm{div}\Pi_{\mathrm{NS}} - \beta \mathbf{g} T, \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathrm{div}(\mathbf{u}T) = \kappa \Delta T. \tag{3}$$

Here, the unknown quantities are the velocity $\mathbf{u}(\mathbf{x}, t)$, the pressure p, and the temperature T. The density of the fluid is assumed to be constant. The viscous stress tensor is defined as

$$\Pi_{\mathrm{NS}} = \nu[(\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^{\mathrm{T}}]. \tag{4}$$

The kinematic viscosity ν, the thermal diffusivity κ, and the coefficient of thermal expansion β are constant and positive; $\mathbf{g}$ is the acceleration due to gravity.

The equations are written using the standard tensor notation. Specifically, $(\mathbf{a} \otimes \mathbf{b})$ is a second-rank tensor invariant obtained as the direct dyadic product of vectors $\mathbf{a}$ and $\mathbf{b}$.

After averaging system (1)–(3) over a short time interval Δt and computing the integral average over the interval $(t, t + \Delta t)$, the *continuity equation* becomes

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} \mathrm{div}\mathbf{u}(t')dt' = \mathrm{div}\left[\frac{1}{\Delta t}\int_t^{t+\Delta t} \mathbf{u}(t')dt'\right] = \mathrm{div}\left[\frac{1}{\Delta t}\mathbf{u}(t^*)\Delta t\right] = \mathrm{div}\mathbf{u}(t^*) = 0. \tag{5}$$

Here, we took into account that spatial differentiation is independent of time integration and used the mean-value theorem to evaluate the time integral. Thus, $\mathbf{u}(t^*)$ is the velocity at some intermediate time t^*, where $t \le t^* \le t + \Delta t$.

Assume that system (1)–(3) has a sufficiently smooth solution. Assuming that the averaging interval is short and the variation in the velocity over the time Δt is small, $\mathbf{u}^*(t)$ can be represented as the first term of the Taylor series expansion:

$$\mathbf{u}^*(t) = \mathbf{u}(t^*) = \mathbf{u}(t) + \tau\frac{\partial \mathbf{u}}{\partial t}. \tag{6}$$

Here, $0 \le \tau \le \Delta t$ is a time smoothing parameter.

The time derivative in (6) is found from the nonconservative form of Eq. (2):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} + \nabla p = \mathrm{div}\Pi_{\mathrm{NS}} - \beta\mathbf{g}T. \tag{7}$$

Omitting the $O(\nu)$ terms, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}\nabla\mathbf{u} - \nabla p - \beta\mathbf{g}T. \tag{8}$$

Thus, the velocity at the intermediate point t^* is represented as

$$\mathbf{u}^* = \mathbf{u} - \mathbf{w}, \tag{9}$$

where

$$\mathbf{w} = \tau(\mathbf{u}\nabla\mathbf{u} + \nabla p + \beta\mathbf{g}T), \tag{10}$$

and the time-averaged continuity equation is

$$\mathrm{div}(\mathbf{u} - \mathbf{w}) = 0. \tag{11}$$

Consider *momentum equation* (2). The first term can be transformed in two ways, specifically,

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} \frac{\partial}{\partial t}\mathbf{u}(t')dt' = \frac{\mathbf{u}(t+\Delta t) - \mathbf{u}(t)}{\Delta t} = \frac{\partial \mathbf{u}}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 \mathbf{u}}{\partial t^2} + O(\Delta t^2) \tag{12}$$

or, using the theorem on the differentiation of an integral with variable limits (see [9]),

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} \frac{\partial}{\partial t}\mathbf{u}(t')dt' = \frac{\partial \mathbf{u}^*}{\partial t} = \frac{\partial}{\partial t}\left(\mathbf{u} + \tau\frac{\partial \mathbf{u}}{\partial t}\right). \tag{13}$$

In both cases,

$$\frac{1}{\Delta t}\int_{t}^{t+\Delta t}\frac{\partial}{\partial t}\mathbf{u}(t')dt' = \frac{\partial \mathbf{u}^*}{\partial t} + O\left(\tau\frac{\partial^2\mathbf{u}}{\partial t^2}\right). \quad (14)$$

The convective term is also averaged over the short time interval and is transformed according to the mean-value theorem with the use of relation (9):

$$\frac{1}{\Delta t}\int_{t}^{t+\Delta t}\operatorname{div}(\mathbf{u}\otimes\mathbf{u})dt' = \operatorname{div}(\mathbf{u}\otimes\mathbf{u})^* = \operatorname{div}(\mathbf{u}(t^*)\otimes\mathbf{u}(t^*)) = \operatorname{div}[(\mathbf{u}-\mathbf{w})\otimes(\mathbf{u}-\mathbf{w})]$$
$$= \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}[(\mathbf{w}\otimes\mathbf{u}) + (\mathbf{u}\otimes\mathbf{w})] + \operatorname{div}(\mathbf{w}\otimes\mathbf{w}). \quad (15)$$

Here, the last term is of order $O(\tau^2)$.

Assume that the variations in pressure and temperature over the averaging interval are negligibly small; i.e., let

$$p(t^*) = p(t), \quad T(t^*) = T(t). \quad (16)$$

While averaging the viscous stress tensor, we retain only linear terms and discard those of order $O(\tau\dot{\nu})$. Then

$$\Pi^*_{\mathrm{NS}} = \Pi_{\mathrm{NS}}.$$

Neglecting the quantity $\sim\tau\partial^2/\partial t^2$ in (14), we obtain the averaged momentum equation

$$\frac{\partial\mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla p = \operatorname{div}\Pi_{\mathrm{NS}} + \operatorname{div}[(\mathbf{w}\otimes\mathbf{u}) + (\mathbf{u}\otimes\mathbf{w})] - \beta\mathbf{g}T. \quad (17)$$

In a similar manner, using assumption (16), we obtain the *smoothed temperature equation*

$$\frac{\partial T}{\partial t} + \operatorname{div}(\mathbf{u}^*T) = \kappa\Delta T. \quad (18)$$

Using the formula for smoothed velocity (9), we obtain

$$\frac{\partial T}{\partial t} + \operatorname{div}(\mathbf{u}T) = \operatorname{div}(\mathbf{w}T) + \kappa\Delta T. \quad (19)$$

Thus, the system of smoothed equations governing viscous incompressible flows has the form

$$\operatorname{div}(\mathbf{u}-\mathbf{w}) = 0, \quad (20)$$

$$\frac{\partial\mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla p = \operatorname{div}\Pi_{\mathrm{NS}} + \operatorname{div}[(\mathbf{w}\otimes\mathbf{u}) + (\mathbf{u}\otimes\mathbf{w})] - \beta\mathbf{g}T, \quad (21)$$

$$\frac{\partial T}{\partial t} + \operatorname{div}(\mathbf{u}T) = \operatorname{div}(\mathbf{w}T) + \kappa\Delta T, \quad (22)$$

where

$$\mathbf{w} = \tau(\mathbf{u}\nabla\mathbf{u} + \nabla p + \beta\mathbf{g}T). \quad (23)$$

On introducing the mass flux density, which is equal to the time-averaged velocity

$$\mathbf{j}_m = \mathbf{u} - \mathbf{w}, \quad (24)$$

system (20)–(23) is rewritten according to [3–6] as the differential conservation laws

$$\operatorname{div}\mathbf{j}_m = 0, \quad (25)$$

$$\frac{\partial\mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{j}_m\otimes\mathbf{u}) + \nabla p = \operatorname{div}\Pi - \beta\mathbf{g}T, \quad (26)$$

$$\frac{\partial T}{\partial t} + \operatorname{div}(\mathbf{j}_m T) = \kappa\Delta T, \quad (27)$$

where the viscous stress tensor is given by

$$\Pi = \Pi_{\mathrm{NS}} + \mathbf{u}\otimes\mathbf{w}. \quad (28)$$

Thus, we have obtained the QHD equations which were previously derived in [3–6].

The following assumptions were used to construct a time-averaged system of equations.

1. The original system has a sufficiently smooth solution.

2. When the mean-value theorem is used, the smoothing parameter τ is the same for all terms and all equations.

3. The quantities of order $O(\tau^2)$, $O(\tau\nu)$, and $O(\tau æ)$ and the terms of the form $\tau\partial^2/\partial t^2$ are small.

4. Only the flow velocity (6) varies over the smoothing interval, while the variations in the pressure and temperature are negligibly small (see (16)).

Smoothed kinetic energy balance equation. Kinetic energy dissipation for system (1)–(3) is described by the equation

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{u}^2}{2}\right) + \operatorname{div}\left[\mathbf{u}\left(\frac{\mathbf{u}^2}{2} + p\right) - \Pi_{\mathrm{NS}}\mathbf{u}\right] = -\Phi \tag{29}$$

with the dissipation function

$$\Phi = \frac{\Pi_{\mathrm{NS}} : \Pi_{\mathrm{NS}}}{2\nu}. \tag{30}$$

By analogy with the above derivation procedure, the smoothed form of Eq. (29) is given by

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{u}^{*2}}{2}\right) + \operatorname{div}\left[\mathbf{u}^*\left(\frac{\mathbf{u}^{*2}}{2} + p\right) - \Pi_{\mathrm{NS}}\mathbf{u}^*\right] = -\Phi. \tag{31}$$

Using $\mathbf{u}^*$ given by (9), we rearrange the left-hand side of (31) and omit the second-order terms to obtain

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{u}^2}{2}\right) + \operatorname{div}\left[(\mathbf{u} - \mathbf{w})\left(\frac{(\mathbf{u} - \mathbf{w})^2}{2} + p\right) - \Pi_{\mathrm{NS}}\mathbf{u}\right] = \frac{\partial}{\partial t}\left(\frac{\mathbf{u}^2}{2}\right) + \operatorname{div}\left[(\mathbf{u} - \mathbf{w})\left(\frac{\mathbf{u}^2}{2} + p\right) - \Pi_{\mathrm{NS}}\mathbf{u} - \mathbf{u}(\mathbf{w}\mathbf{u})\right] = -\Phi. \tag{32}$$

Thus, the time averaging procedure preserves the dissipative character of the equation, but no term of the form $\mathbf{w}^2/\tau$ is added to the dissipation function, which is explained by the lack of rigor of the derivation technique. Such terms were obtained in [3, 6], where the kinetic energy equation was mathematically rigorously derived from the QHD equations for a viscous incompressible fluid.

2. QHD EQUATIONS FOR VISCOUS COMPRESSIBLE GAS FLOWS

The viscous compressible Navier–Stokes equations [9, 10] are written in the index form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i = 0, \tag{33}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}\rho u_i u_j + \frac{\partial}{\partial x_i}p = \rho F_i + \frac{\partial}{\partial x_j}\Pi_{\mathrm{NS}ij}, \tag{34}$$

$$\frac{\partial}{\partial t}\rho\left(\frac{u_i^2}{2} + \varepsilon\right) + \frac{\partial}{\partial x_i}\rho u_i\left(\frac{u_i^2}{2} + \varepsilon + \frac{p}{\rho}\right) + \frac{\partial}{\partial x_i}q_{\mathrm{NS}i} = \rho u_i F_i + \frac{\partial}{\partial x_i}\Pi_{\mathrm{NS}ij}u_j + Q. \tag{35}$$

System (33)–(35) is supplemented with the equations of state

$$p = p(\rho, T), \quad \varepsilon = \varepsilon(\rho, T) \tag{36}$$

and with the following expressions for the viscous stress tensor and the heat flux:

$$\Pi_{\mathrm{NS}ij} = \mu\left(\frac{\partial}{\partial x_i}u_j + \frac{\partial}{\partial x_j}u_i - \frac{2}{3}\delta_{ij}\frac{\partial}{\partial x_k}u_k\right), \quad q_{\mathrm{NS}i} = -k\frac{\partial}{\partial x_i}T. \tag{37}$$

The unknowns in system (33)–(35) are the density ρ, the velocity components u_i, the pressure p, and the internal energy ε. The viscosity μ and the thermal conductivity coefficient k are positive, F_i are the components of the external force, Q is the heat source strength, and δ_{ij} is the Kronecker delta.

The time averaging procedure for this system is similar to that used above. Specifically, we assume that only the flow velocity u_i varies over the averaging interval Δt, while the variations in the density and pressure are negligibly small. Additionally, the variations in the external force and heat source strength are also assumed to be small. In other words, while computing the time averages of the integrals, we set

$$\rho^* = \rho, \quad p^* = p, \quad u_i^* = u_i + \tau\frac{\partial u_i}{\partial t}, \quad F_i^* = F_i, \quad Q^* = Q. \tag{38}$$

The time average of system (33)–(35) is written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i^* = 0, \tag{39}$$

$$\frac{\partial \rho u_i^*}{\partial t} + \frac{\partial}{\partial x_j}\rho(u_i u_j)^* + \frac{\partial}{\partial x_i}p = \rho F_i + \frac{\partial}{\partial x_j}\Pi_{\mathrm{NS}ij}, \tag{40}$$

$$\frac{\partial}{\partial t}\rho\left(\frac{u_i^{*2}}{2}+\varepsilon\right)+\frac{\partial}{\partial x_i}\rho u_i^*\left(\frac{u_i^{*2}}{2}+\varepsilon+\frac{p}{\rho}\right)+\frac{\partial}{\partial x_i}q_{\mathrm{NS}i}=\rho u_i^*F_i+\frac{\partial}{\partial x_i}\Pi_{\mathrm{NS}ij}u_j^*+Q. \tag{41}$$

The velocity u_i^* at the shifted point is computed using (38), where the time derivative is determined from the nonconservative Euler momentum equation

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial}{\partial x_j}u_i+\frac{1}{\rho}\frac{\partial}{\partial x_i}p=F_i. \tag{42}$$

Thus, up to second-order terms in τ, we have

$$u_i^*=u_i+\tau\frac{\partial u_i}{\partial t}=u_i-w_i, \tag{43}$$

where

$$w_i=\frac{\tau}{\rho}\left(\rho u_j\frac{\partial}{\partial x_j}u_i+\frac{\partial}{\partial x_i}p-\rho F_i\right). \tag{44}$$

Introducing the notation for the mass flux density,

$$j_{mi}=\rho(u_i-w_i), \tag{45}$$

we obtain *the time-averaged continuity equation*

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_i}j_{mi}=0. \tag{46}$$

Let us transform *the momentum equation*. The first term is approximately written as

$$\frac{\partial\rho u_i^*}{\partial t}=\frac{\partial}{\partial t}\rho\left(u_i+\tau\frac{\partial u_i}{\partial t}\right)\approx\frac{\partial\rho u_i}{\partial t}. \tag{47}$$

The convective term is transformed as follows:

$$\begin{aligned}\frac{\partial}{\partial x_j}\rho(u_iu_j)^*&=\frac{\partial}{\partial x_j}\rho\left(u_i+\tau\frac{\partial u_i}{\partial t}\right)\left(u_j+\tau\frac{\partial u_j}{\partial t}\right)=\frac{\partial}{\partial x_j}\rho(u_i-w_i)(u_j-w_j)\\&=\frac{\partial}{\partial x_j}\rho(u_iu_j-u_iw_j-u_jw_i+w_iw_j)=\frac{\partial}{\partial x_j}u_jj_{mi}-\frac{\partial}{\partial x_j}\rho u_iw_j.\end{aligned} \tag{48}$$

Here, we discarded the second-order term $\sim\sim w_iw_j$.

Thus, the time-averaged momentum balance equation is

$$\frac{\partial\rho u_i}{\partial t}+\frac{\partial}{\partial x_j}j_{mi}u_j+\frac{\partial}{\partial x_i}p=\rho F_i+\frac{\partial}{\partial x_j}\Pi_{ij}, \tag{49}$$

where the viscous stress tensor is given by

$$\Pi_{ij}=\Pi_{\mathrm{NS}ij}+\rho u_iw_j. \tag{50}$$

Total energy balance equation. By analogy with the previous consideration, the time derivative in the approximation remains unchanged:

$$\frac{\partial}{\partial t}\rho\left(\frac{u_i^{*2}}{2}+\varepsilon\right)\approx\frac{\partial}{\partial t}\rho\left(\frac{u_i^2}{2}+\varepsilon\right). \tag{51}$$

The work of the external forces is written as

$$\rho u_i^*F_i=\rho(u_i-w_i)F_i=j_{mi}F_i. \tag{52}$$

The convective term is rearranged as

$$\begin{aligned}&\frac{\partial}{\partial x_i}\rho u_i^*\left(\frac{u_i^{*2}}{2}+\varepsilon+\frac{p}{\rho}\right)=\frac{\partial}{\partial x_i}\rho(u_i-w_i)\left(\frac{(u_i-w_i)^2}{2}+\varepsilon+\frac{p}{\rho}\right)\\&=\frac{\partial}{\partial x_i}\rho(u_i-w_i)\left(\frac{u_i^2}{2}+\varepsilon+\frac{p}{\rho}\right)+\frac{\partial}{\partial x_i}\rho(u_i-w_i)(-u_iw_i)=\frac{\partial}{\partial x_i}j_{mi}\left(\frac{u_i^2}{2}+\varepsilon+\frac{p}{\rho}\right)-\frac{\partial}{\partial x_i}\rho(u_iu_jw_i)+O(w_iw_j).\end{aligned} \tag{53}$$

Thus, the smoothed energy equation has the form

$$\frac{\partial}{\partial t}\rho\left(\frac{u_i^2}{2}+\varepsilon\right)+\frac{\partial}{\partial x_i}j_{mi}\left(\frac{u_i^2}{2}+\varepsilon+\frac{p}{\rho}\right)+\frac{\partial}{\partial x_i}q_{\mathrm{NS}i}=j_{mi}F_i+\frac{\partial}{\partial x_j}\Pi_{ij}u_j+Q. \tag{54}$$

Angular momentum balance equation. This equation for Navier–Stokes system (33)–(35) has the form

$$\frac{\partial}{\partial t}[\mathbf{x}\times\rho\mathbf{u}] + \operatorname{div}(\rho\mathbf{u}\otimes[\mathbf{x}\times\mathbf{u}]) = [\mathbf{x}\cdot\rho\mathbf{F}] + \frac{\partial}{\partial x_i}[\mathbf{x}\times P_{\mathrm{NS}ij}e_{ij}], \tag{55}$$

where

$$P = -pI + \Pi_{\mathrm{NS}} \tag{56}$$

is the internal stress tensor, $P_{\mathrm{NS}ij}$ denotes the portrait of the tensor P_{NS} in the basis (e_1, e_2, e_3), summation is implied over repeated indices (i, j), and I is the unit tensor. Equation (55) is a consequence of the momentum balance equation (34) in Navier–Stokes system (33)–(35); i.e., it can be derived from the latter with the help of identity transformations.

Equation (55) is averaged over time as follows. By analogy with the above presentation, only the term with the divergence operator is transformed in Eq. (55), specifically,

$$\begin{aligned}\operatorname{div}(\rho\mathbf{u}\otimes[\mathbf{x}\times\mathbf{u}]) &\to \operatorname{div}(\rho\mathbf{u}^*\otimes[\mathbf{x}\times\mathbf{u}^*]) = \operatorname{div}(\rho(\mathbf{u}-\mathbf{w})\otimes[\mathbf{x}\times(\mathbf{u}-\mathbf{w})])\\ &= \operatorname{div}(\mathbf{j}_m\otimes[\mathbf{x}\times\mathbf{u}]) - \operatorname{div}(\rho\mathbf{u}\otimes[\mathbf{x}\times\mathbf{w}]) + O(\mathbf{w}^2).\end{aligned} \tag{57}$$

Thus, the averaged angular momentum balance equation is

$$\frac{\partial}{\partial t}[\mathbf{x}\times\rho\mathbf{u}] + \operatorname{div}(\mathbf{j}_m\otimes[\mathbf{x}\times\mathbf{u}]) = [\mathbf{x}\cdot\rho\mathbf{F}] + \frac{\partial}{\partial x_i}[\mathbf{x}\times P_{ij}e_{ij}], \tag{58}$$

where

$$P = -pI + \Pi_{\mathrm{NS}} + \rho\mathbf{u}\otimes\mathbf{w}. \tag{59}$$

Equation (58) (or the angular momentum theorem) was presented in [3, 6], where it was shown that this equation holds identically under the momentum balance equation (49), as required by the postulates of classical fluid mechanics.

Thus, the system of QHD equations (44)–(46), (49), (50), (54), and (58) with closure (59) is approximately derived by averaging the classical Navier–Stokes equations over time.

Entropy balance equation. The behavior of the entropy

$$s = c_v\ln\left(\frac{RT}{\rho^{\gamma-1}}\right) + \mathrm{const}$$

of system (33)–(35) is described by the equation

$$\frac{\partial\rho s}{\partial t} + \operatorname{div}(\rho\mathbf{u}s) = \operatorname{div}\left(k\frac{\nabla T}{T}\right) + k\left(\frac{\nabla T}{T}\right)^2 + \frac{\Phi}{T} \tag{60}$$

with the dissipation function

$$\Phi = \frac{\Pi_{\mathrm{NS}}:\Pi_{\mathrm{NS}}}{2\mu}. \tag{61}$$

Here, $\Pi_{\mathrm{NS}}:\Pi_{\mathrm{NS}} = \sum_{i,j=1}^{3}(\Pi_{\mathrm{NS}})_{ij}(\Pi_{\mathrm{NS}})_{ij}$ is the double scalar product of two identical tensors, R is the gas constant, γ is the ratio of specific heats, and c_v is the heat capacity of the gas at constant volume.

Formally smoothing this equation over time leads to the velocity in Eq. (60) replaced by its smoothed value:

$$\mathbf{u}\to\mathbf{u}^* = \mathbf{u}-\mathbf{w}. \tag{62}$$

Thus, Eq. (60) becomes

$$\frac{\partial\rho s}{\partial t} + \operatorname{div}(\mathbf{j}_m s) = \operatorname{div}\left(k\frac{\nabla T}{T}\right) + k\left(\frac{\nabla T}{T}\right)^2 + \frac{\Phi}{T}. \tag{63}$$

In contrast to the entropy equation rigorously derived in [3, 6] for the QHD equations, the dissipation function in Eq. (62) lacks the addition $\rho\mathbf{w}^2/\tau$.

3. QGD EQUATIONS FOR VISCOUS COMPRESSIBLE GAS FLOWS

We again average system (33)–(35), but now all the gasdynamic parameters (i.e., the density, velocity, and pressure) are assumed to vary over the short time interval Δt. Proceeding as before, we denote the time averages of quantities by $*$. Then averaging system (33)–(35) yields

$$\frac{\partial\rho^*}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)^* = 0, \tag{64}$$

$$\frac{\partial(\rho u_i)^*}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j)^* + \frac{\partial}{\partial x_i} p^* = (\rho F_i)^* + \frac{\partial}{\partial x_j}\Pi^*_{\mathrm{NS}ij}, \tag{65}$$

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{u_i^2}{2} + \varepsilon\right)\right]^* + \frac{\partial}{\partial x_i}\left[\rho u_i\left(\frac{u_i^2}{2} + \varepsilon + \frac{p}{\rho}\right)\right]^* + \frac{\partial}{\partial x_i} q^*_{\mathrm{NS}i} = (\rho u_i F_i)^* + \frac{\partial}{\partial x_j}\Pi^*_{\mathrm{NS}ij} u_j^* + Q^*. \tag{66}$$

We use the following simplifying assumptions:

$$\rho^* = \rho + \tau\frac{\partial \rho}{\partial t}, \quad p^* = p + \tau\frac{\partial p}{\partial t}, \quad u_i^* = u_i + \tau\frac{\partial u_i}{\partial t} \tag{67}$$

$F_i^* = F_i$, and $Q^* = Q$.

Restricting our consideration to the first-order terms (i.e., neglecting the terms of orders $O(\tau^2)$, $O(\tau\mu)$, and $O(\tau k)$) and omitting terms of the form $\tau\partial^2/\partial t^2$ while computing the time derivatives, we find that system (64)–(66) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho u_i + \tau\frac{\partial}{\partial t}\rho u_i\right) = 0, \tag{68}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}\left[\left(\rho u_i + \tau\frac{\partial}{\partial t}\rho u_i\right)u_j + \tau\rho u_i\frac{\partial}{\partial t}u_j\right] + \frac{\partial}{\partial x_i}\left(p + \tau\frac{\partial}{\partial t}p\right) = \left(\rho + \tau\frac{\partial}{\partial t}\rho\right)F_i + \frac{\partial}{\partial x_j}\Pi_{\mathrm{NS}ij}, \tag{69}$$

$$\begin{aligned}\frac{\partial}{\partial t}\rho\left(\frac{u_i^2}{2} + \varepsilon\right) + \frac{\partial}{\partial x_i}\left[\left(\rho u_i + \tau\frac{\partial}{\partial t}\rho u_i\right)\left(\frac{u_i^2}{2} + \varepsilon + \frac{p}{\rho}\right) + \tau\rho u_i\left(u_j\frac{\partial}{\partial t}u_j + \frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial t}\frac{1}{\rho} + \frac{1}{\rho}\frac{\partial p}{\partial t}\right)\right] \\ + \frac{\partial}{\partial x_i}q_{\mathrm{NS}i} = \left(\rho u_i + \tau\frac{\partial}{\partial t}\rho u_i\right)F_i + \frac{\partial}{\partial x_j}\Pi_{\mathrm{NS}ij}u_j + Q.\end{aligned} \tag{70}$$

Continuity equation (68) is transformed as follows. The time derivative is found from the Euler momentum equation

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial}{\partial x_j}\rho u_i u_j - \frac{\partial}{\partial x_i}p + \rho F_i. \tag{71}$$

Substituting this expression into (68) yields

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left[\rho u_i - \tau\left(\frac{\partial}{\partial x_j}\rho u_i u_j + \frac{\partial}{\partial x_i}p - \rho F_i\right)\right] = 0. \tag{72}$$

Introducing the notation

$$w_i = \frac{\tau}{\rho}\left(\frac{\partial}{\partial x_j}\rho u_i u_j + \frac{\partial}{\partial x_i}p - \rho F_i\right), \quad j_{mi} = \rho(u_i - w_i), \tag{73}$$

we obtain the time-averaged continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}j_{mi} = 0. \tag{74}$$

To transform the momentum and energy equations, we use the Euler equations and differential identities following from them. For an ideal polytropic gas with the equations of state

$$p = \rho R T, \quad \varepsilon = \frac{RT}{\gamma - 1} \tag{75}$$

these identities are

$$\frac{\partial}{\partial t}\frac{1}{\rho} + u_i\frac{\partial}{\partial x_i}\frac{1}{\rho} - \frac{1}{\rho}\frac{\partial}{\partial x_i}u_i = 0, \tag{76}$$

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial}{\partial x_j}u_i + \frac{1}{\rho}\frac{\partial}{\partial x_i}p - F_i = 0, \tag{77}$$

$$\frac{\partial}{\partial t}\varepsilon + u_i\frac{\partial}{\partial x_i}\varepsilon + \frac{p}{\rho}\frac{\partial}{\partial x_i}u_i - \frac{Q}{\rho} = 0, \tag{78}$$

$$\frac{\partial}{\partial t}p + u_i\frac{\partial}{\partial x_i}p + \gamma p\frac{\partial}{\partial x_i}u_i - (\gamma - 1)Q = 0. \tag{79}$$

Let us transform *momentum equation* (69). Using (73) and identities (77) and (79), we obtain

$$\begin{aligned}&\frac{\partial \rho u_i}{\partial t}+\frac{\partial}{\partial x_j}\left[(\rho u_i-\rho w_i)u_j+\tau\rho u_i\left(-u_i\frac{\partial}{\partial x_i}u_j-\frac{1}{\rho}\frac{\partial}{\partial x_j}p+F_j\right)\right]\\&+\frac{\partial}{\partial x_i}\left[p+\tau\left(-u_i\frac{\partial}{\partial x_i}p-\gamma p\frac{\partial}{\partial x_i}u_i+(\gamma-1)Q\right)\right]=\left(\rho-\tau\frac{\partial}{\partial x_i}\rho u_i\right)F_i+\frac{\partial}{\partial x_j}\Pi_{\mathrm{NS}ij}.\end{aligned}\tag{80}$$

Thus, the smoothed momentum equation is written as

$$\frac{\partial \rho u_i}{\partial t}+\frac{\partial}{\partial x_j}j_{mi}u_j+\frac{\partial}{\partial x_i}p=\rho_*F_i+\frac{\partial}{\partial x_j}\Pi_{ij},\tag{81}$$

where the viscous stress tensor is given by

$$\Pi_{ij}=\Pi_{NSij}+\tau\rho u_i\left(u_k\frac{\partial}{\partial x_k}u_j+\frac{1}{\rho}\frac{\partial}{\partial x_j}p-F_j\right)+\tau\delta_{ij}\left(u_k\frac{\partial}{\partial x_k}p+\gamma p\frac{\partial}{\partial x_k}u_k-(\gamma-1)Q\right)\tag{82}$$

and

$$\rho_*=\rho-\tau\frac{\partial}{\partial x_i}\rho u_i.\tag{83}$$

To transform the *total energy balance equation* (70), we use all four Euler identities. As a result,

$$\begin{aligned}&\frac{\partial}{\partial t}\rho\left(\frac{u_i^2}{2}+\varepsilon\right)+\frac{\partial}{\partial x_i}j_{mi}\left(\frac{u_i^2}{2}+\varepsilon+\frac{p}{\rho}\right)+\frac{\partial}{\partial x_i}\tau\rho u_iu_j\left(-u_k\frac{\partial}{\partial x_k}u_j-\frac{1}{\rho}\frac{\partial}{\partial x_i}+F_j\right)\\&-\frac{\partial}{\partial x_i}\tau\rho u_i\frac{1}{\rho}\left(u_k\frac{\partial}{\partial x_k}p+\gamma p\frac{\partial}{\partial x_i}u_i-(\gamma-1)Q\right)+\frac{\partial}{\partial x_i}\tau\rho u_i\left(-u_k\frac{\partial}{\partial x_k}\varepsilon-\frac{p}{\rho}\frac{\partial}{\partial x_k}u_k+\frac{Q}{\rho}\right)\\&+\frac{\partial}{\partial x_i}\tau\rho u_ip\left(-u_k\frac{\partial}{\partial x_k}\frac{1}{\rho}+\frac{1}{\rho}\frac{\partial}{\partial x_k}u_k\right)+\frac{\partial}{\partial x_i}q_{\mathrm{NS}i}=j_{mi}F_i+\frac{\partial}{\partial x_j}\Pi_{\mathrm{NS}ij}u_j+Q.\end{aligned}\tag{84}$$

Combining like terms gives the time-averaged total energy equation

$$\frac{\partial}{\partial t}\rho\left(\frac{u_i^2}{2}+\varepsilon\right)+\frac{\partial}{\partial x_i}j_{mi}\left(\frac{u_i^2}{2}+\varepsilon+\frac{p}{\rho}\right)+\frac{\partial}{\partial x_i}q_i=j_{mi}F_i+\frac{\partial}{\partial x_j}\Pi_{ij}u_j+Q,\tag{85}$$

where the heat flux is given by

$$q_i=q_{\mathrm{NS}i}-\tau\rho u_i\left(u_k\frac{\partial}{\partial x_k}\varepsilon+pu_j\frac{\partial}{\partial x_j}\frac{1}{\rho}-\frac{Q}{\rho}\right).\tag{86}$$

The time-averaged *angular momentum balance equation* is derived in a similar fashion and coincides with the equation obtained in [3, 6] as an exact consequence of the QGD momentum balance equation.

Indeed, under the assumptions made above, smoothed equations (55) and (56) become

$$\frac{\partial}{\partial t}[\mathbf{x}\times\rho\mathbf{u}]+\mathrm{div}(\rho\mathbf{u}\otimes[\mathbf{x}\times\mathbf{u}])^*=[\mathbf{x}\cdot\rho\mathbf{F}]^*+\frac{\partial}{\partial x_i}[\mathbf{x}\times P^*_{\mathrm{NS}ij}e_{ij}],\tag{87}$$

where

$$P^*_{\mathrm{NS}}=-p^*I+\Pi_{\mathrm{NS}}=-pI+\tau I[(\mathbf{u}\cdot\nabla)p+\gamma p\,\mathrm{div}\mathbf{u}-(\gamma-1)Q]+\Pi_{\mathrm{NS}},\tag{88}$$

$$[\mathbf{x}\cdot\rho\mathbf{F}]^*=[\mathbf{x}\cdot\rho^*\mathbf{F}]=[\mathbf{x}\cdot(\rho-\tau\mathrm{div}(\rho\mathbf{u}))\mathbf{F}].\tag{89}$$

The convective term in (87) is transformed using the relations

$$\begin{aligned}(\rho\mathbf{u}\otimes[\mathbf{x}\times\mathbf{u}])^*&=(\rho\mathbf{u})^*\otimes[\mathbf{x}\times\mathbf{u}^*]=\mathbf{j}_m\otimes[\mathbf{x}\times\mathbf{u}]+\rho\mathbf{u}\otimes[\mathbf{x}\times\mathbf{u}^*]\\&=\mathbf{j}_m\otimes[\mathbf{x}\times\mathbf{u}]-\tau\mathbf{u}\otimes[\mathbf{x}\times[\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\rho\mathbf{F}]].\end{aligned}\tag{90}$$

Thus, the smoothed momentum balance equation has the form

$$\frac{\partial}{\partial t}[\mathbf{x}\times\rho\mathbf{u}]+\mathrm{div}(\mathbf{j}_m\otimes[\mathbf{x}\times\mathbf{u}])=[\mathbf{x}\cdot\rho_*\mathbf{F}]+\frac{\partial}{\partial x_i}[\mathbf{x}\times P_{ij}e_{ij}],\tag{91}$$

where

$$P=-pI+\Pi_{\mathrm{NS}}+\tau\mathbf{u}\otimes[\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\rho\mathbf{F}]+\tau I[(\mathbf{u}\cdot\nabla)p+\gamma p\,\mathrm{div}\mathbf{u}-(\gamma-1)Q].\tag{92}$$

The smoothed Navier–Stokes equations thus constructed coincide with the QGD equations obtained in [3–6] by different approximate methods.

According to [11], the QGD equations (73), (74), (81)–(83), (85), and (86) can be extended to gases with the general equation of state (36) by making the substitutions

$$\gamma p \to \rho c_s^2, \quad \gamma - 1 \to \gamma_Q - 1, \tag{93}$$

in the last Euler identity (79), which leads to the corresponding substitution in the second bracket of (82). The speed of sound and the ratio of specific heats in (93) are calculated as

$$c_s^2 = \frac{\partial p}{\partial \rho} + \frac{T(\partial p/\partial T)^2}{\rho^2 \partial \varepsilon/\partial T}, \quad \gamma_Q = \frac{\partial p/\partial T}{\rho \partial \varepsilon/\partial T} + 1. \tag{94}$$

To conclude, the dissipation function in the entropy balance equation (60) for the QGD system with external forces and heat sources (see [5]) is given by

$$\Phi = \frac{\Pi_{NSij} : \Pi_{NSij}}{2\mu} + \tau\rho\left(u_k \frac{\partial}{\partial x_k} u_i + \frac{1}{\rho}\frac{\partial}{\partial x_i} p - F_i\right)^2 + \frac{\tau\rho}{\varepsilon}\left(u_i \frac{\partial}{\partial x_i}\varepsilon + \frac{p}{\rho}\frac{\partial}{\partial x_i} u_i - \frac{Q}{2\rho}\right)^2 + Q\left(1 - \tau\frac{(\gamma-1)Q}{4p}\right). \tag{95}$$

As in the previous case, we fail to obtain terms $\sim\tau$ in (95) by approximately averaging the classical entropy equation over time. Such terms are obtained when an entropy balance equation of form (60) is derived directly from the QGD equations; i.e., the operations of averaging and deriving the entropy equation on the basis of the continuity, momentum, and energy equations are not commutative.

CONCLUSIONS

A new approximate method for constructing the QGD and QHD equations was presented that is based on the smoothing of the classical Navier–Stokes equations over a short time interval. The method is closely related to the approximate procedure for deriving the QGD and QHD systems [3–6] consisting of the integration of the classical equations over a small stationary volume and the subsequent estimation of the fluxes crossing its boundary.

The method can be used to construct the continuity, momentum, angular momentum, and energy balance equations for the QHD and QGD systems. However, the dissipation function in the resulting entropy balance equation for the QHD and QGD systems does not involve additional terms $\sim\tau$, which are inherent in the these systems. This result is explained by the lack of rigor of the method for constructing the QHD and QGD systems.

The new method for constructing the QGD and QHD systems is more transparent and less cumbersome than the previously used techniques. Moreover, all the versions of these systems, including the QGD and QHD equations for viscous incompressible flows, can be derived in a unified manner. The present approach can be used to obtain a whole family of smoothed QGD- and QHD-type systems based on various fluid dynamics equations. Specifically, the above method was used in [12] to derive smoothed shallow water equations. In [13] this method was applied to obtain new versions of the QHD equations for compressible plasma flows in an electromagnetic field, including the averaging of a magnetic field. The first versions of such systems were presented in [6]. It is of interest to develop this approach to two-fluid magnetohydrodynamics.

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