A SHALLOW WATERS MODEL WITH A NEW VISCOSITY TERM BY ASYMPTOTIC ANALYSIS

José M. Rodríguez* and Raquel Taboada-Vázquez†

*Departmento de Métodos Matemáticos e de Representación.
E. T. S. Arquitectura. Universidade da Coruña
Campus da Zapateira s/n. 15071 - A Coruña. Spain.
e-mail: mmrseijo@udc.es

† Departmento de Métodos Matemáticos e de Representación.
E. T. S. Náutica e Máquinas. Universidade da Coruña
Paseo de Ronda, 51. 15011 - A Coruña. Spain
e-mail: raqueltv@udc.es

Key words: Asymptotic analysis, shallow waters with viscosity.

Abstract. In this paper, we study the Navier-Stokes equations in a domain with small depth. With this aim, we introduce a small adimensional parameter \( \varepsilon \) related to the depth. First we make a change of variable to a domain independent of \( \varepsilon \) and then we use asymptotic analysis to study what happens when \( \varepsilon \) becomes small. This way we obtain a model for \( \varepsilon \) small that, after coming back to the original domain and without making a priori assumptions about velocity or pressure behaviour, gives us a shallow water model including a new diffusion term.
1 INTRODUCTION

In this article we shall apply asymptotic analysis with the aim of obtaining a shallow water model with a new viscosity term. To achieve this, we suppose that, in the original domain, flow obeys Navier-Stokes equations and we introduce a small adimensional parameter $\varepsilon$ related to the depth. Next, we make a change of variable to a domain independent of $\varepsilon$ and we suppose that the solution of Navier-Stokes equations in the new domain allows an expansion in powers of $\varepsilon$. When $\varepsilon$ is small we can keep just the first terms of the expansion and obtain a valid model for the case of small depth. With these terms we build an approximation that, after coming back to the original domain, gives us a shallow water model including a new diffusion term.

Usually, when using asymptotic analysis to analyze fluids, we use it in the original domain (see, for example, [1] and [2]), that in this case depends on parameter $\varepsilon$ and time $t^\varepsilon$, or the surface is supposed to be constant (see, for example, [3]). However, we have preferred to use the asymptotic technique in the same way as in [4], [5], [6], [7] and related works, that is, we make a change of variable to a reference domain independent of the parameter $\varepsilon$ and the time.

In order to locate the problem, we begin considering a river or a region of the sea represented by a domain $\Omega^\varepsilon$ (see Fig. 1) defined by

$$\Omega^\varepsilon = \{(x^\varepsilon, y^\varepsilon, z^\varepsilon)/ (x^\varepsilon, y^\varepsilon) \in D, z^\varepsilon \in (H^\varepsilon(x^\varepsilon, y^\varepsilon), s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon))\}$$

(1)

![Figure 1: Domain](image)

where $x^\varepsilon$ and $y^\varepsilon$ are the horizontal coordinates, $z^\varepsilon$ is the vertical coordinate, $D$ is the projection over the XY plane of $\Omega^\varepsilon$, $z^\varepsilon = H^\varepsilon(x^\varepsilon, y^\varepsilon)$ is the equation of the bottom of the river or the sea (supposed known), $z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)$ is the equation of the surface (unknown). We can also define $h^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) - H^\varepsilon(x^\varepsilon, y^\varepsilon)$ (water depth).
We want to obtain a shallow water model, so depth must be small if compared with the width and length of the domain. With this aim we define an adimensional small parameter $\varepsilon$, such that it represents the quotient between characteristic depth and length of the domain.

We can now suppose that $H^\varepsilon(x^\varepsilon, y^\varepsilon) = \varepsilon H(x, y)$, $s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon s(t, x, y)$ and $h^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon h(t, x, y)$, (where $x = x^\varepsilon$, $y = y^\varepsilon$ and $t = t^\varepsilon$ are independent of $\varepsilon$).

Let us consider that flow obeys three dimensional Navier-Stokes equations in $\Omega^\varepsilon$ and that the external forces acting over the fluid are those due to gravity and the Coriolis acceleration, that is
\[
\begin{align*}
\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon + w^\varepsilon \partial_z u^\varepsilon &= -(\rho_0)^{-1} \partial^\varepsilon_p + \nu \Delta^\varepsilon u^\varepsilon + 2\phi (\sin^\varepsilon \nu^\varepsilon - \cos^\varepsilon \nu^\varepsilon) \quad (2) \\
\partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon + w^\varepsilon \partial_z v^\varepsilon &= -(\rho_0)^{-1} \partial^\varepsilon_p + \nu \Delta^\varepsilon v^\varepsilon - 2\phi \sin^\varepsilon \nu^\varepsilon \quad (3) \\
\partial_t w^\varepsilon + u^\varepsilon \partial_x w^\varepsilon + v^\varepsilon \partial_y w^\varepsilon + w^\varepsilon \partial_z w^\varepsilon &= -(\rho_0)^{-1} \partial^\varepsilon_p - g + \nu \Delta^\varepsilon w^\varepsilon + 2\phi \cos^\varepsilon \nu^\varepsilon \quad (4)
\end{align*}
\]
where $\partial^\varepsilon_p = \partial/\partial x^\varepsilon$, $\partial_x = \partial/\partial x^\varepsilon$, $\partial_y = \partial/\partial y^\varepsilon$, $\partial_z = \partial/\partial z^\varepsilon$, $\Delta_x^\varepsilon = \partial^2/\partial(x^\varepsilon)^2$, $\Delta_y^\varepsilon = \partial^2/\partial(y^\varepsilon)^2$, $\Delta_z^\varepsilon = \partial^2/\partial(z^\varepsilon)^2$, $\Delta^\varepsilon = \Delta_x^\varepsilon + \Delta_y^\varepsilon + \Delta_z^\varepsilon$, $\rho_0$ is the density of the fluid, $\nu$ is the kinematic viscosity coefficient ($\nu = \mu/\rho_0$ where $\mu$ is the first viscosity coefficient) and $g$ is the gravity acceleration (supposed constant). The Coriolis acceleration $-2\vec{\omega}\times\vec{v}$ has been calculated as in [2], using that the angular velocity of rotation of the Earth is $\vec{\omega} = \phi \vec{\varphi}$ with $\phi = 7.29 \times 10^{-5}$ rad/s and $\vec{\varphi} = \sin^\varepsilon \vec{i} + \cos^\varepsilon \vec{j}$ where $\vec{i}$, $\vec{j}$ and $\vec{k}$ denote the unit vectors pointing East, North and vertically upward (respectively) and $\varphi^\varepsilon$ is the North latitude (that we consider either constant or depending on $y^\varepsilon$).

The fluid is supposed to be incompressible, so it verifies
\[
\partial_x^\varepsilon u^\varepsilon + \partial_y^\varepsilon v^\varepsilon + \partial_z^\varepsilon w^\varepsilon = 0
\]
Let us introduce now the boundary conditions. The pressure must be the atmospheric on the surface and the fluid can not penetrate the bottom so,
\[
\begin{align*}
p^\varepsilon &= p_s^\varepsilon \quad \text{on} \quad z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) \quad (6) \\
(u^\varepsilon, v^\varepsilon, w^\varepsilon) \cdot \vec{n}^\varepsilon &= 0 \quad \text{on} \quad z^\varepsilon = H^\varepsilon(x^\varepsilon, y^\varepsilon) \quad (7)
\end{align*}
\]
where $\vec{n}^\varepsilon$ is the outward unit normal vector to the boundary. In addition we need to know the behaviour of the velocity on the boundary $\partial D \times (H^\varepsilon, s^\varepsilon)$. We impose a condition of non penetration (as in (7)) all over $\partial D \times (H^\varepsilon, s^\varepsilon)$ except on the open sections of the region, where the flow must be known.

To take into account the effects of the wind on the surface and the friction on the bottom we must introduce the stress tensor
\[
T^\varepsilon = \begin{pmatrix} T_{11}^\varepsilon & T_{12}^\varepsilon & T_{13}^\varepsilon \\ T_{21}^\varepsilon & T_{22}^\varepsilon & T_{23}^\varepsilon \\ T_{31}^\varepsilon & T_{32}^\varepsilon & T_{33}^\varepsilon \end{pmatrix} = \mu \begin{pmatrix} 2 \partial_x^\varepsilon u^\varepsilon & \partial_y^\varepsilon u^\varepsilon + \partial_z^\varepsilon v^\varepsilon & \partial_z^\varepsilon u^\varepsilon + \partial_x^\varepsilon w^\varepsilon \\ \partial_y^\varepsilon u^\varepsilon + \partial_z^\varepsilon v^\varepsilon & 2 \partial_y^\varepsilon v^\varepsilon + \partial_x^\varepsilon w^\varepsilon & \partial_z^\varepsilon v^\varepsilon + \partial_y^\varepsilon w^\varepsilon \\ \partial_x^\varepsilon u^\varepsilon + \partial_z^\varepsilon w^\varepsilon & \partial_z^\varepsilon v^\varepsilon + \partial_y^\varepsilon w^\varepsilon & 2 \partial_z^\varepsilon w^\varepsilon \end{pmatrix}
\]
\]

José M. Rodríguez and Raquel Taboada-Vázquez
and then both effects can be written
\[
(T^\varepsilon \cdot \vec{n}^\varepsilon)_\tau = \vec{f}_W^\varepsilon \quad \text{on} \quad z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) \tag{9}
\]
\[
(T^\varepsilon \cdot \vec{n}^\varepsilon)_\tau = -\vec{f}_R^\varepsilon \quad \text{on} \quad z^\varepsilon = H^\varepsilon(x^\varepsilon, y^\varepsilon) \tag{10}
\]
where \((T^\varepsilon \cdot \vec{n}^\varepsilon)_\tau\) is the projection of \(T^\varepsilon \cdot \vec{n}^\varepsilon\) on the tangent plane to the boundary and \(\vec{f}_W^\varepsilon = (\vec{f}_{W1}^\varepsilon, \vec{f}_{W2}^\varepsilon)\), \(\vec{f}_R^\varepsilon = (\vec{f}_{R1}^\varepsilon, \vec{f}_{R2}^\varepsilon)\) are the force of the wind and the friction force, respectively, that can be computed using empirical expressions found in literature (see, for example, [8] or [9]). We shall suppose that condition (10) is also verified on the non-open sections of \(\partial D \times (H^\varepsilon, s^\varepsilon)\), where we are going to consider that \(\vec{f}_R^\varepsilon = \vec{0}\).

Classically, to obtain an equation that determines \(h^\varepsilon\), it is used a free surface condition on \(z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)\), but we have used an equation derived from the mass conservation law:
\[
\partial_t^\varepsilon h^\varepsilon + \partial_x^\varepsilon \int_{H^\varepsilon}^z u^\varepsilon \, dz^\varepsilon + \partial_y^\varepsilon \int_{H^\varepsilon}^z v^\varepsilon \, dz^\varepsilon = 0 \tag{11}
\]
To close the equations we only need to set the initial conditions for \(t^\varepsilon = 0\).

Combining (2)-(4) with (5) and (8) we obtain the following equations
\[
\begin{align*}
\partial_t^\varepsilon u^\varepsilon + u^\varepsilon \partial_x^\varepsilon u^\varepsilon + v^\varepsilon \partial_y^\varepsilon u^\varepsilon + w^\varepsilon \partial_z^\varepsilon u^\varepsilon &= -\left(\rho_0\right)^{-1} \partial_x^\varepsilon p^\varepsilon + 2\phi \left(\sin \phi^\varepsilon v^\varepsilon - \cos \phi^\varepsilon w^\varepsilon\right) \\
&+ \nu \left\{ 2 \partial_x^2 u^\varepsilon + \partial_y^2 u^\varepsilon + \partial_z^2 u^\varepsilon \right\} + \left(\rho_0\right)^{-1} \partial_x^\varepsilon T_{13}^\varepsilon \tag{12}
\end{align*}
\]
\[
\begin{align*}
\partial_t^\varepsilon v^\varepsilon + u^\varepsilon \partial_x^\varepsilon v^\varepsilon + v^\varepsilon \partial_y^\varepsilon v^\varepsilon + w^\varepsilon \partial_z^\varepsilon v^\varepsilon &= -\left(\rho_0\right)^{-1} \partial_y^\varepsilon p^\varepsilon - 2\phi \sin \phi^\varepsilon u^\varepsilon \\
&+ \nu \left\{ 2 \partial_y^2 v^\varepsilon + \partial_x^2 v^\varepsilon + 2 \partial_z^2 v^\varepsilon \right\} + \left(\rho_0\right)^{-1} \partial_y^\varepsilon T_{23}^\varepsilon \tag{13}
\end{align*}
\]
\[
\begin{align*}
\partial_t^\varepsilon w^\varepsilon + u^\varepsilon \partial_x^\varepsilon w^\varepsilon + v^\varepsilon \partial_y^\varepsilon w^\varepsilon + w^\varepsilon \partial_z^\varepsilon w^\varepsilon &= -\left(\rho_0\right)^{-1} \partial_z^\varepsilon p^\varepsilon - g + 2\phi \cos \phi^\varepsilon u^\varepsilon \\
&+ 2\nu \partial_z^2 w^\varepsilon + \left(\rho_0\right)^{-1} \left( \partial_x^\varepsilon T_{13}^\varepsilon + \partial_y^\varepsilon T_{23}^\varepsilon \right) \tag{14}
\end{align*}
\]
that we shall use instead of (2)-(4) because they will allow us to incorporate to limit equations the effects of wind and friction.

2 CONSTRUCTION OF THE REFERENCE DOMAIN

Our purpose in this paper is to obtain a new shallow waters model taking as starting point the Navier-Stokes equations and using asymptotic analysis. In general, when used asymptotics to analyze fluids, they are used in the original domain, that in this case depends on parameter \(\varepsilon\) and time \(t^\varepsilon\), or the surface is supposed to be constant. If we work in a domain independent of \(\varepsilon\), all the “dependency” of this parameter appears explicit in the equations, whereas when we work with domains that depend on \(\varepsilon\), part of this dependency can remain “hidden” in the domain. That is the reason why we prefer to follow the usual techniques in asymptotic analysis applied to solids (see [4], [5], [6], [7] and the references indicated in them) in our study of the shallow waters. That is, we begin making a change of variable to a reference domain independent of the parameter \(\varepsilon\) and the time.
Let $\Omega = D \times (0,1)$ be the reference domain and let us define the following change of variable, from $\Omega$ to $\Omega^\varepsilon$:

$$t^\varepsilon = t, \quad x^\varepsilon = x, \quad y^\varepsilon = y, \quad z^\varepsilon = \varepsilon[H(x,y) + zh(t,x,y)]$$ (15)

Given any function $F^\varepsilon$ defined on $[0,T] \times \overline{\Omega}$, we can define other function $F(\varepsilon)$ on $[0,T] \times \overline{\Omega}$ using the change of variable: $F(\varepsilon)(t,x,y,z) = F^\varepsilon(t^\varepsilon,x^\varepsilon,y^\varepsilon,z^\varepsilon)$. The relationship between the partial derivatives is the following:

$$\partial_t F^\varepsilon = \partial_t F(\varepsilon) - \frac{z}{h} \partial_t h \partial_z F(\varepsilon) = D_t F(\varepsilon),$$

$$\partial_x F^\varepsilon = \partial_x F(\varepsilon) - \frac{\partial_x H + z \partial_z h}{h} \partial_z F(\varepsilon) = D_x F(\varepsilon),$$

$$\partial_y F^\varepsilon = \partial_y F(\varepsilon) - \frac{\partial_y H + z \partial_y h}{h} \partial_z F(\varepsilon) = D_y F(\varepsilon),$$

$$\partial_z F^\varepsilon = \frac{1}{\varepsilon h} \partial_z F(\varepsilon) = \frac{1}{\varepsilon} D_z F(\varepsilon),$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, $\partial_z = \partial/\partial z$, and where we introduce the following notation:

$$D_t = \partial_t - \frac{z}{h} \partial_t h \partial_z, \quad D_x = \partial_x - \frac{\partial_x H + z \partial_z h}{h} \partial_z, \quad D_y = \partial_y - \frac{\partial_y H + z \partial_y h}{h} \partial_z,$$

$$D_z = \frac{1}{h} \partial_z, \quad D_x^2 = D_x(D_x), \quad D_y^2 = D_y(D_y), \quad D_{xy}^2 = D_x(D_y), \quad D_z^2 = D_z(D_z).$$

Applying this change of variable to $u^\varepsilon$, $v^\varepsilon$, $w^\varepsilon$, $p^\varepsilon$ and $T_{ij}^\varepsilon$ $(i,j = 1,2,3)$, we obtain

$$u(\varepsilon)(t,x,y,z) = u^\varepsilon(t^\varepsilon,x^\varepsilon,y^\varepsilon,z^\varepsilon), \quad v(\varepsilon)(t,x,y,z) = v^\varepsilon(t^\varepsilon,x^\varepsilon,y^\varepsilon,z^\varepsilon),$$

$$w(\varepsilon)(t,x,y,z) = w^\varepsilon(t^\varepsilon,x^\varepsilon,y^\varepsilon,z^\varepsilon), \quad p(\varepsilon)(t,x,y,z) = p^\varepsilon(t^\varepsilon,x^\varepsilon,y^\varepsilon,z^\varepsilon),$$

$$T_{ij}(\varepsilon)(t,x,y,z) = T_{ij}^\varepsilon(t^\varepsilon,x^\varepsilon,y^\varepsilon,z^\varepsilon), \quad (i,j = 1,2,3).$$ (16)

We also apply the change of variable to equations (5)-(14), so the modified Navier-Stokes equations, the incompressibility equation and the mass conservation law on $\Omega$ can be written as follows, where now dependence on $\varepsilon$ appears explicitly, (we suppose that $\varphi^\varepsilon = \varphi$ is independent of $\varepsilon$).

$$D_t u(\varepsilon) + u(\varepsilon)D_x u(\varepsilon) + v(\varepsilon)D_y u(\varepsilon) + w(\varepsilon)D_z u(\varepsilon) + \frac{1}{\varepsilon} D_z p(\varepsilon) + 2\phi (\sin \varphi v(\varepsilon) - \cos \varphi w(\varepsilon)) + \nu \left\{ {2D_x^2 u(\varepsilon) + D_y^2 u(\varepsilon) + D_{xy}^2 v(\varepsilon)} \right\} + \frac{1}{\rho_0 \varepsilon} D_z T_{13}(\varepsilon)$$ (17)


\[ D_t v(\varepsilon) + u(\varepsilon) D_x v(\varepsilon) + v(\varepsilon) D_y v(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z v(\varepsilon) = -\frac{1}{\rho_0} D_y p(\varepsilon) - 2\phi \sin \varphi u(\varepsilon) \]

\[ + \nu \left\{ D_{xy}^2 u(\varepsilon) + D_x^2 v(\varepsilon) + 2D_y^2 v(\varepsilon) \right\} + \frac{1}{\rho_0} \frac{1}{\varepsilon} D_z T_{23}(\varepsilon) \]  
(18)

\[ D_t w(\varepsilon) + u(\varepsilon) D_x w(\varepsilon) + v(\varepsilon) D_y w(\varepsilon) + w(\varepsilon) \frac{1}{\varepsilon} D_z w(\varepsilon) = -\frac{1}{\rho_0} \frac{1}{\varepsilon} D_z p(\varepsilon) + 2\phi \cos \varphi u(\varepsilon) \]

\[ - g + 2\nu \frac{1}{\varepsilon^2} D_x^2 w(\varepsilon) + \frac{1}{\rho_0} \left\{ D_z T_{13}(\varepsilon) + D_y T_{23}(\varepsilon) \right\} \]  
(19)

\[ D_x u(\varepsilon) + D_y v(\varepsilon) + \frac{1}{\varepsilon} D_z w(\varepsilon) = 0 \]  
(20)

\[ \partial_t h + \int_0^1 \left[ \partial_x (u(\varepsilon) h) + \partial_y (v(\varepsilon) h) \right] dz = 0 \]  
(21)

The change of variable is applied to the initial and boundary conditions too.

### 3 ASYMPTOTIC ANALYSIS

Let us suppose now that the solution of the problem (17)-(21) allows an expansion in powers of \( \varepsilon \), that is,

\[ u(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots, \quad v(\varepsilon) = v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots, \]
\[ w(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \cdots, \quad p(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots, \]
\[ T_{ij}(\varepsilon) = T_{ij}^{(0)} + \varepsilon T_{ij}^{(1)} + \varepsilon^2 T_{ij}^{(2)} + \cdots \quad (i, j = 1, 2), \]
\[ T_{ij}(\varepsilon) = \varepsilon^{-1} T_{ij}^{(1)} + \varepsilon T_{ij}^{(1)} + \varepsilon^2 T_{ij}^{(2)} + \cdots \quad (i = 1, 2, 3). \]  
(22)

We replace this expansion into the equations obtained, after the change of variable, on \( \Omega \). Making this substitution in (17), we get:

\[ D_t u^0 + \varepsilon D_t u^1 + \varepsilon^2 D_t u^2 + \cdots + \left( u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots \right) \left[ D_x u^0 + \varepsilon D_x u^1 + \varepsilon^2 D_x u^2 + \cdots \right] \]

\[ + \left( v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots \right) \left[ D_y u^0 + \varepsilon D_y u^1 + \varepsilon^2 D_y u^2 + \cdots \right] \]

\[ + \left( w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots \right) \frac{1}{\varepsilon} \left[ D_z u^0 + \varepsilon D_z u^1 + \varepsilon^2 D_z u^2 + \cdots \right] \]

\[ = -\frac{1}{\rho_0} \left[ D_x p^0 + \varepsilon D_x p^1 + \varepsilon^2 D_x p^2 + \cdots \right] \]

\[ + 2\phi \left[ \sin \varphi (v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots) - \cos \varphi (w^0 + \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \cdots) \right] \]

\[ + \nu \left\{ D_{xy}^2 u^0 + \varepsilon D_{xy}^2 u^1 + \varepsilon^2 D_{xy}^2 u^2 + \cdots \right\} + D_y u^0 + \varepsilon D_y u^1 + \varepsilon^2 D_y u^2 + \cdots \]

\[ + D_{xy} v^0 + \varepsilon D_{xy} v^1 + \varepsilon^2 D_{xy} v^2 + \cdots \]

\[ + \frac{1}{\rho_0} \frac{1}{\varepsilon} \left[ \varepsilon^{-1} D_z T_{13}^{(1)} + D_z T_{13}^{(2)} + \varepsilon D_z T_{13}^{(3)} + \varepsilon^2 D_z T_{13}^{(4)} + \varepsilon^3 D_z T_{13}^{(5)} + \cdots \right] \]  
(23)
Similar expressions are obtained for the other equations.

Next step is to identify terms multiplied by the same power of \( \varepsilon \). Let see how (23) can be rewritten grouping terms:

\[
-\varepsilon^{-2} \frac{1}{\rho_0} D_z T_{13}^{-1} + \varepsilon^{-1} \left\{ w^0 D_z u^0 - \frac{1}{\rho_0} D_z T_{13} \right\} \\
+ D_t u^0 + u^0 D_x u^0 + v^0 D_y u^0 + w^0 D_z u^0 + w^1 D_z u^0 + \frac{1}{\rho_0} D_x p^0 \\
- 2 \phi \left[ \sin \varphi v^0 - \cos \varphi w^0 \right] - \nu \left[ 2 D_x^2 u^0 + D_y^2 u^0 + D_{xy}^2 v^0 \right] - \frac{1}{\rho_0} D_z T_{13} \\
+ \varepsilon \left\{ D_t u^1 + u^0 D_x u^1 + u^1 D_x u^0 + v^0 D_y u^1 + v^1 D_y u^0 + w^0 D_z u^2 + w^1 D_z u^1 + w^2 D_z u^0 \right\} \\
+ \frac{1}{\rho_0} D_x p^1 - 2 \phi \left[ \sin \varphi v^1 - \cos \varphi w^1 \right] - \nu \left[ 2 D_x^2 u^1 + D_y^2 u^1 + D_{xy}^2 v^1 \right] - \frac{1}{\rho_0} D_z T_{13} \\
+ \varepsilon^2 \left\{ D_t u^2 + u^0 D_x u^2 + u^1 D_x u^1 + u^2 D_x u^0 + v^0 D_y u^2 + v^1 D_y u^1 + v^2 D_y u^0 \right\} \\
+ w^0 D_z u^3 + w^1 D_z u^2 + w^2 D_z u^1 + w^3 D_z u^0 + \frac{1}{\rho_0} D_x p^2 - 2 \phi \left[ \sin \varphi v^2 - \cos \varphi w^2 \right] \\
- \nu \left[ 2 D_x^2 u^2 + D_y^2 u^2 + D_{xy}^2 v^2 \right] - \frac{1}{\rho_0} D_z T_{13} + \cdots = 0
\] (24)

Replacing the expansions in powers of \( \varepsilon \) (22) in (18) and then grouping terms multiplied by the same power of \( \varepsilon \) we obtain:

\[
-\varepsilon^{-2} \frac{1}{\rho_0} D_z T_{23}^{-1} + \varepsilon^{-1} \left\{ w^0 D_z v^0 - \frac{1}{\rho_0} D_z T_{23} \right\} \\
+ D_t v^0 + u^0 D_x v^0 + v^0 D_y v^0 + w^0 D_z v^0 + w^1 D_z v^0 + \frac{1}{\rho_0} D_y p^0 \\
+ 2 \phi \sin \varphi u^0 - \nu \left[ D_x^2 v^0 + D_y^2 v^0 + 2 D_{xy}^2 v^0 \right] - \frac{1}{\rho_0} D_z T_{23} \\
+ \varepsilon \left\{ D_t v^1 + u^0 D_x v^1 + u^1 D_x v^0 + v^0 D_y v^1 + v^1 D_y v^0 + w^0 D_z v^2 + w^1 D_z v^1 \right\} \\
+ w^2 D_z v^0 + \frac{1}{\rho_0} D_y p^1 + 2 \phi \sin \varphi u^1 - \nu \left[ D_x^2 v^1 + D_y^2 v^1 + 2 D_{xy}^2 v^1 \right] - \frac{1}{\rho_0} D_z T_{23} \\
+ \varepsilon^2 \left\{ D_t v^2 + u^0 D_x v^2 + u^1 D_x v^1 + u^2 D_x v^0 + v^0 D_y v^2 + v^1 D_y v^1 + v^2 D_y v^0 \right\} \\
+ w^0 D_z v^3 + w^1 D_z v^2 + w^2 D_z v^1 + w^3 D_z v^0 + \frac{1}{\rho_0} D_y p^2 + 2 \phi \sin \varphi u^2 \\
- \nu \left[ D_x^2 v^2 + D_y^2 v^2 + 2 D_{xy}^2 v^2 \right] - \frac{1}{\rho_0} D_z T_{23} + \cdots = 0
\] (25)

We repeat the process for (19),

\[
-\varepsilon^{-2} 2 \nu D_z^2 w^0 + \varepsilon^{-1} \left\{ w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 - 2 \nu D_z^2 w^1 - \frac{1}{\rho_0} D_z T_{13} - D_y T_{23} \right\}
\]
+ D_t w^0 + u^0 D_x w^0 + v^0 D_y w^0 + w^0 D_z w^0 + w^1 D_z w^0 + \frac{1}{\rho_0} D_z p^1 \\
- 2\phi \cos \varphi u^0 + g - 2\nu D^2 w^2 - \frac{1}{\rho_0} \left[ D_x T^0_{13} + D_y T^0_{23} \right] \\
+ \varepsilon \left\{ D_t w^1 + u^0 D_x w^1 + u^1 D_x w^0 + v^0 D_y w^1 + v^1 D_y w^0 + w^0 D_z w^2 + w^1 D_z w^1 \right. \\
+ \left. w^2 D_z w^0 + \frac{1}{\rho_0} D_z p^2 - 2\phi \cos \varphi u^1 - 2\nu D^2 w^3 - \frac{1}{\rho_0} \left[ D_x T^1_{13} + D_y T^1_{23} \right] \right\} + \cdots = 0 \\n(26)

We do the same with (20) so we obtain:

\[ \varepsilon^{-1} D_t w^0 + D_x u^0 + D_y v^0 + D_z w^1 + \varepsilon \left( D_x u^1 + D_y v^1 + D_z w^2 \right) \]

\[ + \varepsilon^2 \left( D_x u^2 + D_y v^2 + D_z w^3 \right) + \cdots = 0 \] \\
(27)

From the boundary conditions (6)-(7) (we suppose \( p_s^0 = p_s \) independent of \( \varepsilon \)) we have:

\[ p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \cdots = p_s \quad \text{on } z = 1 \] \\
(28)

\[ -w^0 + \varepsilon \left\{ u^0 \partial_x H + v^0 \partial_y H - w^1 \right\} + \varepsilon^2 \left\{ u^1 \partial_x H + v^1 \partial_y H - w^2 \right\} \]

\[ + \varepsilon^3 \left\{ u^2 \partial_x H + v^2 \partial_y H - w^3 \right\} + \cdots = 0 \quad \text{on } z = 0 \] \\
(29)

Making an asymptotic analysis of the most common empirical expressions for the force of the wind and the friction force (see [8] or [9]) we can suppose the following expansions:

\[ f(\varepsilon)_{R_i} = \varepsilon f^1_{R_i} + \varepsilon^2 f^2_{R_i} + \cdots \quad f(\varepsilon)_{W_i} = \varepsilon f^1_{W_i} + \varepsilon^2 f^2_{W_i} + \cdots \quad (i = 1, 2) \]

(30)

Considering the previous assumptions, replacing the expansions in powers of \( \varepsilon \) in (9) and grouping terms multiplied by the same power of \( \varepsilon \) we obtain:

\[ -\varepsilon^{-1} T^0_{13} - \left\{ T^0_{13} + \partial_x H T^0_{33} \right\} + \varepsilon \left\{ \partial_x H T^0_{11} + \partial_y H T^0_{12} - T^0_{13} \right. \]

\[ + \left. \partial_x H \left[ \partial_x H T^{-1}_{13} + \partial_y H T^{-1}_{23} - T^{-1}_{33} \right] + f^1_{R_i} \right\} + \cdots = 0 \quad \text{on } z = 0 \] \\
(31)

We obtain a similar equation on \( z = 0 \) for the second component of the friction force and two more on \( z = 1 \) for the force of the wind.

Starting off from (21), the equation necessary to calculate the depth, we get the expression:

\[ \partial_t h + \int_0^1 \left[ \partial_x (u^0 h) + \partial_y (v^0 h) \right] dz + \varepsilon \int_0^1 \left[ \partial_x (u^1 h) + \partial_y (v^1 h) \right] dz \]

\[ + \varepsilon^2 \int_0^1 \left[ \partial_x (u^2 h) + \partial_y (v^2 h) \right] dz + \cdots = 0 \] \\
(32)
Now, equaling the coefficients of each power of $\varepsilon$ to zero, we obtain a series of equations that are used to determine $u^0$, $v^0$, $w^0$, $p^0$, $u^1$, $v^1$, etc.

We begin with the coefficients of $\varepsilon^{-2}$:

$$D_z T_{13}^{-1} = 0, \quad D_z T_{23}^{-1} = 0, \quad D_z^2 w^0 = 0.$$  \hspace{1cm} (33)

Equaling the coefficients of $\varepsilon^{-1}$ of each equation to zero we obtain:

$$w^0 D_z v^0 - \frac{1}{\rho_0} D_z T_{13}^0 = 0$$  \hspace{1cm} (34)

$$w^0 D_z w^0 - \frac{1}{\rho_0} D_z T_{23}^0 = 0$$  \hspace{1cm} (35)

$$w^0 D_z w^0 + \frac{1}{\rho_0} D_z p^0 - 2\nu D_z^2 w^1 - \frac{1}{\rho_0} \left[ D_z T_{13}^{-1} + D_z T_{23}^{-1} \right] = 0$$  \hspace{1cm} (36)

$$D_z w^0 = 0$$  \hspace{1cm} (37)

$$T_{13}^{-1} = T_{23}^{-1} = 0 \quad \text{on} \quad z = 0$$  \hspace{1cm} (38)

$$T_{13}^{-1} = T_{23}^{-1} = 0 \quad \text{on} \quad z = 1$$  \hspace{1cm} (39)

As consequence of (8), (33.a), (33.b), (37) and (38) we know that:

$$T_{13}^{-1} = T_{23}^{-1} = T_{33}^{-1} = 0$$  \hspace{1cm} (40)

$$\partial_z u^0 = \partial_z v^0 = 0$$  \hspace{1cm} (41)

what means that $u^0$ and $v^0$ are independent of $z$.

Next, we equal the coefficients of $\varepsilon^0$ to 0 and considering (33)-(41), we obtain:

$$\partial_t u^0 + u^0 \partial_x u^0 + v^0 \partial_y u^0 + w^0 D_z u^1 + \frac{1}{\rho_0} D_z p^0$$

$$- 2\phi \left[ \sin \varphi u^0 - \cos \varphi w^0 \right] - \nu \left[ \partial_x^2 u^0 + \partial_y^2 u^0 + \partial_{xy}^2 u^0 \right] - \frac{1}{\rho_0} D_z T_{13}^1 = 0$$  \hspace{1cm} (42)

$$\partial_t v^0 + u^0 \partial_x v^0 + v^0 \partial_y v^0 + w^0 D_z v^1 + \frac{1}{\rho_0} D_y p^0$$

$$+ 2\phi \sin \varphi u^0 - \nu \left[ \partial_{xy}^2 u^0 + \partial_x^2 v^0 + 2\partial_y^2 v^0 \right] - \frac{1}{\rho_0} D_z T_{23}^1 = 0$$  \hspace{1cm} (43)

$$D_t w^0 + u^0 D_z w^0 + v^0 D_y w^0 + w^0 D_z w^1 + \frac{1}{\rho_0} D_z p^1$$

$$- 2\phi \cos \varphi u^0 + g - 2\nu D_z^2 w^2 - \frac{1}{\rho_0} \left[ D_z T_{13}^0 + D_z T_{23}^0 \right] = 0$$  \hspace{1cm} (44)

$$\partial_x u^0 + \partial_y v^0 + D_z w^1 = 0$$  \hspace{1cm} (45)

$$p^0 = p_s \quad \text{on} \quad z = 1$$  \hspace{1cm} (46)
\[ w^0 = 0 \quad \text{on} \quad z = 0 \quad (47) \]
\[ T_{13}^0 = T_{23}^0 = 0 \quad \text{on} \quad z = 0 \quad (48) \]
\[ T_{13}^0 = T_{23}^0 \quad \text{on} \quad z = 1 \quad (49) \]
\[ \partial_t h + \partial_x (u^0 h) + \partial_y (v^0 h) = 0 \quad (50) \]

Using (5), (8), (34), (35), (36), (37) and (40), from the equalities (42)-(50) we can draw the following consequences:

\[ w^0 = 0 \quad (51) \]
\[ T_{13}^0 = T_{23}^0 = 0 \quad (52) \]
\[ \partial_z u^1 = \partial_z v^1 = 0 \quad (53) \]
\[ p^0 = p_s \quad (54) \]
\[ \frac{1}{\rho_0} D_z p^1 = 2\phi \cos \varphi u^0 - g \quad (55) \]

\((u^1, v^1 \text{ and } p^0 \text{ do not depend on } z \text{ either})\).

We can simplify (42) and (43) using (51) and (54). The result is:

\[ \partial_t u^0 + u^0 \partial_x u^0 + v^0 \partial_y u^0 = -\frac{1}{\rho_0} \partial_x p_s + 2\phi \sin \varphi v^0 \]
\[ + \nu \left[ 2\partial_x^2 u^0 + \partial_y^2 u^0 + \partial_{xy}^2 v^0 \right] + \frac{1}{\rho_0} D_z T_{13}^1 \quad (56) \]
\[ \partial_t v^0 + u^0 \partial_x v^0 + v^0 \partial_y v^0 = -\frac{1}{\rho_0} \partial_y p_s - 2\phi \sin \varphi u^0 \]
\[ + \nu \left[ \partial_{xy}^2 u^0 + \partial_z^2 v^0 + 2\partial_y^2 v^0 \right] + \frac{1}{\rho_0} D_z T_{23}^1 \quad (57) \]

where \(u^0, v^0\) and \(p_s\) do not depend on \(z\).

Equating the coefficients of \(e^1\) of each equation to 0 and bearing in mind (33)-(55) we obtain:

\[ \partial_t u^1 + u^0 \partial_x u^1 + u^1 \partial_x u^0 + v^0 \partial_y u^1 + v^1 \partial_y u^0 + \frac{1}{\rho_0} D_z p^1 \]
\[ - 2\phi \left[ \sin \varphi v^1 - \cos \varphi w^1 \right] - \nu \left[ 2\partial_x^2 u^1 + \partial_y^2 u^1 + \partial_{xy}^2 v^1 \right] - \frac{1}{\rho_0} D_z T_{13}^2 = 0 \quad (58) \]
\[ \partial_t v^1 + u^0 \partial_x v^1 + u^1 \partial_x v^0 + v^0 \partial_y v^1 + v^1 \partial_y v^0 + \frac{1}{\rho_0} D_y p^1 \]
\[ + 2\phi \sin \varphi u^1 - \nu \left[ \partial_{xy}^2 u^1 + \partial_z^2 v^1 + 2\partial_y^2 v^1 \right] - \frac{1}{\rho_0} D_z T_{23}^2 = 0 \quad (59) \]
\[ D_t w^1 + u^0 D_x w^1 + v^0 D_y w^1 + w^1 D_z w^1 + \frac{1}{\rho_0} D_z p^2 \]

10
\[-2\phi \cos \varphi u^1 - 2\nu D_z^2 w^3 - \frac{1}{\rho_0} \left[D_x T_{13}^1 + D_y T_{23}^1 \right] = 0 \] (60)

\[\partial_x u^1 + \partial_y v^1 + D_z w^2 = 0 \] (61)

\[p^1 = 0 \quad \text{on } z = 1 \] (62)

\[w^1 = u^0 \partial_x H + v^0 \partial_y H \quad \text{on } z = 0 \] (63)

\[T_{13}^1 = \partial_x H \quad T_{11}^0 + \partial_y H \quad T_{12}^0 - \partial_z H \quad T_{33}^0 + f_{R1}^1 \quad \text{on } z = 0 \] (64)

\[T_{23}^1 = \partial_x H \quad T_{12}^0 + \partial_y H \quad T_{22}^0 - \partial_z H \quad T_{33}^0 + f_{R2}^1 \quad \text{on } z = 0 \] (65)

\[T_{13}^1 = \partial_z s \quad T_{11}^0 + \partial_y s \quad T_{12}^0 - \partial_x s \quad T_{33}^0 + f_{W1}^1 \quad \text{on } z = 1 \] (66)

\[T_{23}^1 = \partial_z s \quad T_{12}^0 + \partial_y s \quad T_{22}^0 - \partial_x s \quad T_{33}^0 + f_{W2}^1 \quad \text{on } z = 1 \] (67)

The process continues with the coefficients of \(\varepsilon^2, \varepsilon^3, \ldots \).

Now, we integrate (55) imposing condition (62), it gives us the following expression for \(p^1\):

\[p^1 = \rho_0 h(z - 1) \left(2\phi \cos \varphi u^0 - g \right) \] (68)

We integrate (45) respect to \(z\), considering that \(u^0\) and \(v^0\) do not depend on \(z\). Then, imposing (63) we can find \(w^1\):

\[w^1 = u^0 \partial_x H + v^0 \partial_y H - zh \left\{\partial_x u^0 + \partial_y v^0\right\} \] (69)

In a similar way we obtain:

\[w^2 = u^1 \partial_x H + v^1 \partial_y H - zh \left\{\partial_x u^1 + \partial_y v^1\right\} \] (70)

We can use expressions (68) and (69) to replace \(p^1\) and \(w^1\) respectively in the equations (58)-(59) so the dependency on \(z\) of these equations is explicit except in the terms \(D_x T_{33}^2 \quad (i = 1, 2)\).

4 FIRST ORDER APPROXIMATION

We consider the following approximation:

\[\hat{u}(\varepsilon) = u^0 + \varepsilon u^1, \quad \hat{v}(\varepsilon) = v^0 + \varepsilon v^1, \quad \hat{w}(\varepsilon) = w^0 + \varepsilon w^1 + \varepsilon^2 w^2, \quad \hat{p}(\varepsilon) = p^0 + \varepsilon p^1, \]

\[\hat{T}_{ij}(\varepsilon) = T_{ij}^0 + \varepsilon T_{ij}^1 \quad (i, j = 1, 2), \quad \hat{T}_{i3}(\varepsilon) = \varepsilon^{-1} T_{i3}^{-1} + T_{i3}^0 + \varepsilon T_{i3}^1 \quad (i = 1, 2, 3) \] (71)

and we do the change of variable back to \(\Omega^\varepsilon\), then the associated functions on \(\Omega^\varepsilon\)

\[\tilde{u}(\varepsilon)(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \hat{u}(\varepsilon)(t, x, y, z), \quad \tilde{v}(\varepsilon)(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \hat{v}(\varepsilon)(t, x, y, z), \quad \tilde{w}(\varepsilon)(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \hat{w}(\varepsilon)(t, x, y, z), \quad \tilde{p}(\varepsilon)(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \hat{p}(\varepsilon)(t, x, y, z) \] (72)

verify

\[\partial_z \tilde{u}(\varepsilon) = 0 , \] (73)
\[
\frac{\partial \varepsilon}{\partial t} \tilde{v}^\varepsilon = 0, \\
\tilde{w}^\varepsilon = \tilde{u} \frac{\partial \varepsilon}{\partial t} H^\varepsilon + \tilde{v} \frac{\partial \varepsilon}{\partial y} H^\varepsilon - (\tilde{z}^\varepsilon - H^\varepsilon)[\frac{\partial \varepsilon}{\partial x} \tilde{u}^\varepsilon + \frac{\partial \varepsilon}{\partial y} \tilde{v}^\varepsilon], \\
\tilde{p}^\varepsilon = p_s + \rho_0 (\tilde{z}^\varepsilon - s^\varepsilon) (2\phi \cos \varphi \tilde{u}^\varepsilon - g) + O(\varepsilon^2), \\
\frac{\partial \varepsilon}{\partial t} h^\varepsilon + \frac{\partial \varepsilon}{\partial x} (\tilde{u}^\varepsilon h^\varepsilon) + \frac{\partial \varepsilon}{\partial y} (\tilde{v}^\varepsilon h^\varepsilon) = O(\varepsilon^2), \\
\frac{\partial \varepsilon}{\partial t} \tilde{u}^\varepsilon + \tilde{u} \frac{\partial \varepsilon}{\partial x} \tilde{u}^\varepsilon + \tilde{v} \frac{\partial \varepsilon}{\partial y} \tilde{u}^\varepsilon - \nu \left\{ \Delta_{xy} \tilde{u}^\varepsilon + (\varepsilon^\varepsilon) - 2 \partial_y \tilde{u}^\varepsilon + \partial_y \tilde{v}^\varepsilon \right\} = - \frac{\partial \varepsilon}{\partial x} \tilde{p}^\varepsilon \\
- g \partial_s^\varepsilon + 2 \phi \cos \varphi \tilde{v}^\varepsilon + 2 \phi \cos \varphi \tilde{u}^\varepsilon \left\{ \partial_x^\varepsilon \tilde{u}^\varepsilon + \partial_y^\varepsilon \tilde{u}^\varepsilon - \tilde{v} \partial_y^\varepsilon H^\varepsilon + \frac{1}{2} \tilde{v}^\varepsilon \partial_y^\varepsilon \tilde{v}^\varepsilon \right\} \\
+ \left( \rho_0 \varepsilon \right) - 1 \left( f_{W1}^\varepsilon - f_{R1}^\varepsilon \right) + O(\varepsilon^2) \\
\tilde{v} \frac{\partial \varepsilon}{\partial t} \tilde{v}^\varepsilon + \tilde{u} \frac{\partial \varepsilon}{\partial x} \tilde{v}^\varepsilon + \tilde{v} \frac{\partial \varepsilon}{\partial y} \tilde{v}^\varepsilon - \nu \left\{ \Delta_{xy} \tilde{v}^\varepsilon + (\varepsilon^\varepsilon) - 2 \partial_y \tilde{u}^\varepsilon + \partial_y \tilde{v}^\varepsilon \right\} = - \frac{\partial \varepsilon}{\partial y} \tilde{p}^\varepsilon \\
- g \partial_s^\varepsilon - 2 \phi \sin \varphi \tilde{u}^\varepsilon + 2 \phi \cos \varphi \tilde{v}^\varepsilon \left\{ \partial_x^\varepsilon \tilde{u}^\varepsilon + \partial_y^\varepsilon \tilde{u}^\varepsilon - \tilde{v} \partial_y^\varepsilon H^\varepsilon + \frac{1}{2} \tilde{v}^\varepsilon \partial_y^\varepsilon \tilde{v}^\varepsilon \right\} \\
+ \left( \rho_0 \varepsilon \right) - 1 \left( f_{W2}^\varepsilon - f_{R2}^\varepsilon \right) + O(\varepsilon^2). 
\]

5 PROPOSED MODEL

If in (73)-(79) we neglect the terms in \(O(\varepsilon^2)\), we obtain the following shallow water model (where we drop the \(\tilde{\cdot}\) for the sake of clarity) whose order of precision, at least formally, is \(O(\varepsilon)\):

\[
\frac{\partial \varepsilon}{\partial t} h^\varepsilon + \frac{\partial \varepsilon}{\partial x} (u^\varepsilon h^\varepsilon) + \frac{\partial \varepsilon}{\partial y} (v^\varepsilon h^\varepsilon) = 0 \\
\frac{\partial \varepsilon}{\partial t} u^\varepsilon + u^\varepsilon \frac{\partial \varepsilon}{\partial x} u^\varepsilon + v^\varepsilon \frac{\partial \varepsilon}{\partial y} u^\varepsilon - \nu \left\{ \Delta_{xy} u^\varepsilon + 2 \frac{\partial \varepsilon}{\partial x} (u^\varepsilon h^\varepsilon) \right\} \\
+ (\varepsilon^\varepsilon) - 2 \frac{\partial \varepsilon}{\partial x} (u^\varepsilon h^\varepsilon) + v^\varepsilon \frac{\partial \varepsilon}{\partial y} u^\varepsilon = - \frac{\partial \varepsilon}{\partial x} \tilde{p}^\varepsilon \\
- g \partial_s^\varepsilon + 2 \phi \cos \varphi \tilde{v}^\varepsilon - 2 \phi \cos \varphi \tilde{u}^\varepsilon \left\{ \partial_x^\varepsilon u^\varepsilon + \partial_y^\varepsilon u^\varepsilon \right\} + \frac{1}{2} \tilde{v}^\varepsilon \partial_y^\varepsilon \tilde{v}^\varepsilon \\
+ \left( \rho_0 \varepsilon \right) - 1 \left( f_{W1}^\varepsilon - f_{R1}^\varepsilon \right) \]

\[
\frac{\partial \varepsilon}{\partial t} v^\varepsilon + u^\varepsilon \frac{\partial \varepsilon}{\partial x} v^\varepsilon + v^\varepsilon \frac{\partial \varepsilon}{\partial y} v^\varepsilon - \nu \left\{ \Delta_{xy} v^\varepsilon + 2 \frac{\partial \varepsilon}{\partial y} (u^\varepsilon h^\varepsilon) \right\} \\
+ (\varepsilon^\varepsilon) - 2 \frac{\partial \varepsilon}{\partial y} (u^\varepsilon h^\varepsilon) + v^\varepsilon \frac{\partial \varepsilon}{\partial y} v^\varepsilon = - \frac{\partial \varepsilon}{\partial y} \tilde{p}^\varepsilon \\
- g \partial_s^\varepsilon - 2 \phi \sin \varphi \tilde{u}^\varepsilon + 2 \phi \cos \varphi \tilde{v}^\varepsilon \left\{ \partial_x^\varepsilon u^\varepsilon + \partial_y^\varepsilon u^\varepsilon \right\} + \frac{1}{2} \tilde{v}^\varepsilon \partial_y^\varepsilon \tilde{v}^\varepsilon \\
+ \left( \rho_0 \varepsilon \right) - 1 \left( f_{W2}^\varepsilon - f_{R2}^\varepsilon \right) \]

\[
p^\varepsilon = p_s + \rho_0 (\tilde{z}^\varepsilon - s^\varepsilon) (2 \phi \cos \varphi \tilde{u}^\varepsilon - g), \\
u^\varepsilon = u^\varepsilon \frac{\partial \varepsilon}{\partial x} H^\varepsilon + v^\varepsilon \frac{\partial \varepsilon}{\partial y} H^\varepsilon - (\tilde{z}^\varepsilon - H^\varepsilon) \left[ \frac{\partial \varepsilon}{\partial x} \tilde{u}^\varepsilon + \frac{\partial \varepsilon}{\partial y} \tilde{v}^\varepsilon \right]
\]

(remember that \(u^\varepsilon\), \(v^\varepsilon\) and \(h^\varepsilon\) do not depend on \(z^\varepsilon\)) and where the initial and boundary conditions at \(\partial D\), as well as \(p_s^\varepsilon\), are supposed to be known.
Let us remark that the diffusion terms in (81) and (82) are new and then, the model (80)-(84) supposes an interesting modification of other shallow water models. Moreover, we have obtained this model without making a priori assumptions about pressure or velocity, and we have also obtained a non zero vertical velocity \( \varepsilon \) which takes into account the effects of a non constant bottom.

If we incorporate the second order terms in (71), the model improves its approximation order formally but the expressions obtained are much more complicated. For example, if we consider the approximation of the pressure \( \bar{p}(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 \), then we obtain the following correction of \( \bar{p} \):

\[
\bar{p}^\varepsilon = p_0 - \rho_0 \varphi \cos \varphi (s^\varepsilon - z^\varepsilon) u^\varepsilon
+ \rho_0 (s^\varepsilon - z^\varepsilon) \left\{ g + \partial_t \left[ u^\varepsilon \partial_x H^\varepsilon + v^\varepsilon \partial_y H^\varepsilon \right] + u^\varepsilon \partial_t \left[ u^\varepsilon \partial_x H^\varepsilon + v^\varepsilon \partial_y H^\varepsilon \right] + v^\varepsilon \partial_y \left[ u^\varepsilon \partial_x H^\varepsilon + v^\varepsilon \partial_y H^\varepsilon \right] - \frac{\rho_0}{2} (h^\varepsilon)^2 - (z^\varepsilon - H^\varepsilon)^2 \left\{ \partial_x^\varepsilon \left[ \partial_x^\varepsilon u^\varepsilon + \partial_y^\varepsilon v^\varepsilon \right] - \left[ \partial_x^\varepsilon u^\varepsilon + \partial_y^\varepsilon v^\varepsilon \right]^2 \right\} \right\}.
\]

In order to compare the model proposed with different models that we have found in the literature, so we can observe what has of novel, we rewrite model (80)-(84) in dimension one (we neglect Coriolis force because in dimension one it makes no sense):

\[
\begin{align*}
\partial_t^\varepsilon h^\varepsilon + \partial_x^\varepsilon (u^\varepsilon h^\varepsilon) &= 0, \\
\partial_t^\varepsilon u^\varepsilon + u^\varepsilon \partial_x^\varepsilon u^\varepsilon - 2\nu (h^\varepsilon)^{-2} \partial_x^\varepsilon \left\{ (h^\varepsilon)^2 \partial_x^\varepsilon u^\varepsilon \right\} &= -(\rho_0)^{-1} \partial_x^\varepsilon p_s^\varepsilon - g \partial_x^\varepsilon s^\varepsilon + (\rho_0 h^\varepsilon)^{-1} (f_{\bar{W}} - f_R^\varepsilon), \\
p^\varepsilon &= p_s^\varepsilon + \rho_0 g (s^\varepsilon - z^\varepsilon), \\
w^\varepsilon &= u^\varepsilon \partial_x^\varepsilon H^\varepsilon - (z^\varepsilon - H^\varepsilon) \partial_x^\varepsilon u^\varepsilon.
\end{align*}
\]

Next, we shall write all the other models including the effects of wind and friction although some of them do not consider those effects, with the aim of better compare them with our model.

In first place, the classic shallow water model without viscosity (see for example [8], page 38 or [11], page 3)

\[
\begin{align*}
\partial_t^\varepsilon h^\varepsilon + \partial_x^\varepsilon (u^\varepsilon h^\varepsilon) &= 0, \\
\partial_t^\varepsilon u^\varepsilon + u^\varepsilon \partial_x^\varepsilon u^\varepsilon &= -(\rho_0)^{-1} \partial_x^\varepsilon p_s^\varepsilon - g \partial_x^\varepsilon s^\varepsilon + (\rho_0 h^\varepsilon)^{-1} (f_{\bar{W}} - f_R^\varepsilon).
\end{align*}
\]

Secondly, we can find models like [12] (page 60) that introduce viscosity in a similar way to the Navier-Stokes equations:

\[
\begin{align*}
\partial_t^\varepsilon h^\varepsilon + \partial_x^\varepsilon (u^\varepsilon h^\varepsilon) &= 0, \\
\partial_t^\varepsilon u^\varepsilon + u^\varepsilon \partial_x^\varepsilon u^\varepsilon - \nu \partial_x^2 s^\varepsilon &= -(\rho_0)^{-1} \partial_x^\varepsilon p_s^\varepsilon - g \partial_x^\varepsilon s^\varepsilon + (\rho_0 h^\varepsilon)^{-1} (f_{\bar{W}} - f_R^\varepsilon).
\end{align*}
\]

José M. Rodríguez and Raquel Taboada-Vázquez
There are models that add another viscosity term (see for example [13], page 1137 or [14], page 302). This sort of models can be written as follows:

\[ \partial_t^\varepsilon h^\varepsilon + \partial_x^\varepsilon (u^\varepsilon h^\varepsilon) = 0 \]  
\[ \partial_t^\varepsilon u^\varepsilon + u^\varepsilon \partial_x^\varepsilon u^\varepsilon - \nu (h^\varepsilon)^{-1} \partial_x^\varepsilon (h^\varepsilon \partial_x^\varepsilon u^\varepsilon) = - (\rho_0)^{-1} \partial_x^\varepsilon p^\varepsilon - g \partial_x^\varepsilon s^\varepsilon \]  
\[ + (\rho_0 h^\varepsilon)^{-1} (f_W - f_R) \]  
(95)

If we expand the viscosity term of (95):

\[ \nu (h^\varepsilon)^{-1} \partial_x^\varepsilon (h^\varepsilon \partial_x^\varepsilon u^\varepsilon) = \nu \left\{ \partial_x^{2,\varepsilon} u^\varepsilon + (h^\varepsilon)^{-1} \partial_x^\varepsilon h^\varepsilon \partial_x^\varepsilon u^\varepsilon \right\} \]

we can appreciate that the model adds to the “Navier-Stokes” viscosity the term

\[ \nu (h^\varepsilon)^{-1} \partial_x^\varepsilon h^\varepsilon \partial_x^\varepsilon u^\varepsilon. \]

Finally, if we rewrite the viscosity term of our model (86)-(89)

\[ 2\nu (h^\varepsilon)^{-2} \partial_x^\varepsilon ((h^\varepsilon)^2 \partial_x^\varepsilon u^\varepsilon) = \nu \left\{ 2\partial_x^{2,\varepsilon} u^\varepsilon + 4(h^\varepsilon)^{-1} \partial_x^\varepsilon h^\varepsilon \partial_x^\varepsilon u^\varepsilon \right\} \]

we find that in this case the effect of the “Navier-Stokes” viscosity is duplicated while the new diffusion term seen in (95) is multiplied by 4.

6 FROM EULER TO SHALLOW WATERS BY ASYMPTOTIC ANALYSIS

What happens when we use the Euler equations as starting point instead of Navier-Stokes equations?

If we consider that the flow obeys three dimensional Euler equations in \( \Omega^\varepsilon \), we have:

\[ \partial_t^\varepsilon u^\varepsilon + u^\varepsilon \partial_x^\varepsilon u^\varepsilon + v^\varepsilon \partial_y^\varepsilon u^\varepsilon + w^\varepsilon \partial_z^\varepsilon u^\varepsilon = - (\rho_0)^{-1} \partial_x^\varepsilon p^\varepsilon + 2\phi (\sin \varphi^\varepsilon v^\varepsilon - \cos \varphi^\varepsilon w^\varepsilon) \]  
(96)

\[ \partial_t^\varepsilon v^\varepsilon + u^\varepsilon \partial_x^\varepsilon v^\varepsilon + v^\varepsilon \partial_y^\varepsilon v^\varepsilon + w^\varepsilon \partial_z^\varepsilon v^\varepsilon = - (\rho_0)^{-1} \partial_y^\varepsilon p^\varepsilon - 2\phi \sin \varphi^\varepsilon u^\varepsilon \]  
(97)

\[ \partial_t^\varepsilon w^\varepsilon + u^\varepsilon \partial_x^\varepsilon w^\varepsilon + v^\varepsilon \partial_y^\varepsilon w^\varepsilon + w^\varepsilon \partial_z^\varepsilon w^\varepsilon = - (\rho_0)^{-1} \partial_z^\varepsilon p^\varepsilon - g + 2\phi \cos \varphi^\varepsilon u^\varepsilon \]  
(98)

When we apply asymptotic analysis (after making the change of variable to the reference domain defined in (15)), we shall need more equations. In the simplest case of dimension one (see [10]) it is enough to suppose that vorticity is zero at the initial time and, so, at any time. Nevertheless, in this case, this assumption is not realistic, because we are considering non conservative forces. This is the reason why we use the vorticity equations:

\[ \partial_t^\varepsilon \gamma_1^\varepsilon + u^\varepsilon \partial_x^\varepsilon \gamma_1^\varepsilon + v^\varepsilon \partial_y^\varepsilon \gamma_1^\varepsilon + w^\varepsilon \partial_z^\varepsilon \gamma_1^\varepsilon - \gamma_1^\varepsilon \partial_x^\varepsilon u^\varepsilon - \gamma_2^\varepsilon \partial_y^\varepsilon u^\varepsilon - \gamma_3^\varepsilon \partial_z^\varepsilon u^\varepsilon = \]

\[ = 2\phi \left( \partial_y^\varepsilon (\cos \varphi^\varepsilon u^\varepsilon) + \sin \varphi^\varepsilon \partial_x^\varepsilon u^\varepsilon \right) \]  
(99)

\[ \partial_t^\varepsilon \gamma_2^\varepsilon + u^\varepsilon \partial_x^\varepsilon \gamma_2^\varepsilon + v^\varepsilon \partial_y^\varepsilon \gamma_2^\varepsilon + w^\varepsilon \partial_z^\varepsilon \gamma_2^\varepsilon - \gamma_1^\varepsilon \partial_x^\varepsilon v^\varepsilon - \gamma_2^\varepsilon \partial_y^\varepsilon v^\varepsilon - \gamma_3^\varepsilon \partial_z^\varepsilon v^\varepsilon = \]

\[ = 2\phi \left( \sin \varphi^\varepsilon \partial_x^\varepsilon v^\varepsilon + \cos \varphi^\varepsilon \partial_y^\varepsilon v^\varepsilon \right) \]  
(100)

\[ \partial_t^\varepsilon \gamma_3^\varepsilon + u^\varepsilon \partial_x^\varepsilon \gamma_3^\varepsilon + v^\varepsilon \partial_y^\varepsilon \gamma_3^\varepsilon + w^\varepsilon \partial_z^\varepsilon \gamma_3^\varepsilon - \gamma_1^\varepsilon \partial_x^\varepsilon w^\varepsilon - \gamma_2^\varepsilon \partial_y^\varepsilon w^\varepsilon - \gamma_3^\varepsilon \partial_z^\varepsilon w^\varepsilon = \]

\[ = 2\phi \left( - \sin \varphi^\varepsilon \partial_x^\varepsilon w^\varepsilon - \partial_y^\varepsilon (\sin \varphi^\varepsilon v^\varepsilon) + \partial_z^\varepsilon (\cos \varphi^\varepsilon w^\varepsilon) \right) \]  
(101)
where:
\[ \gamma_1^\varepsilon = \partial_y^\varepsilon \hat{w}^\varepsilon - \partial_z^\varepsilon \hat{v}^\varepsilon \quad \gamma_2^\varepsilon = \partial_z^\varepsilon \hat{u}^\varepsilon - \partial_x^\varepsilon \hat{w}^\varepsilon \quad \gamma_3^\varepsilon = \partial_x^\varepsilon \hat{v}^\varepsilon - \partial_y^\varepsilon \hat{u}^\varepsilon \] (102)

are the vorticity vector components.

In this way we obtain the following shallow water model whose precision order is \( O(\varepsilon^2) \) too:

\[ \partial_t^\varepsilon \hat{u}^\varepsilon + \hat{u}^\varepsilon \partial_x^\varepsilon \hat{u}^\varepsilon + \hat{v}^\varepsilon (\partial_y^\varepsilon \hat{u}^\varepsilon - 2\phi \sin \varphi^\varepsilon) = -(\rho_0)^{-1} \partial_x^\varepsilon p^\varepsilon + 2\phi \cos \varphi^\varepsilon \partial_x^\varepsilon \hat{u}^\varepsilon h^\varepsilon \]
\[ + \partial_x^\varepsilon s^\varepsilon (2\phi \cos \varphi^\varepsilon \hat{r}^\varepsilon - g) - 2\phi \cos \varphi^\varepsilon (\hat{u}^\varepsilon \partial_y^\varepsilon H^\varepsilon + \hat{v}^\varepsilon \partial_y^\varepsilon H^\varepsilon) \] (103)

\[ \partial_t^\varepsilon \hat{v}^\varepsilon + \hat{u}^\varepsilon (\partial_x^\varepsilon \hat{v}^\varepsilon + 2\phi \sin \varphi^\varepsilon) + \hat{v}^\varepsilon \partial_y^\varepsilon \hat{v}^\varepsilon = -(\rho_0)^{-1} \partial_y^\varepsilon p^\varepsilon + 2\phi \partial_y^\varepsilon (\cos \varphi^\varepsilon \hat{u}^\varepsilon) h^\varepsilon \]
\[ + \partial_y^\varepsilon s^\varepsilon (2\phi \cos \varphi^\varepsilon \hat{r}^\varepsilon - g) \] (104)

\[ \partial_t^\varepsilon \hat{r}^\varepsilon + \hat{u}^\varepsilon \partial_x^\varepsilon \hat{r}^\varepsilon + \hat{v}^\varepsilon (\partial_y^\varepsilon \hat{r}^\varepsilon + 2\phi \sin \varphi^\varepsilon) + \hat{v}^\varepsilon \partial_y^\varepsilon \hat{r}^\varepsilon = -2\phi \left\{ \frac{\gamma_1^\varepsilon \sin \varphi^\varepsilon - \cos \varphi^\varepsilon \partial_y^\varepsilon \hat{v}^\varepsilon}{\gamma_1^\varepsilon} \right\} \] (105)

\[ \partial_t^\varepsilon \gamma_1^\varepsilon + \hat{u}^\varepsilon \partial_x^\varepsilon \gamma_1^\varepsilon + \hat{v}^\varepsilon \partial_y^\varepsilon \gamma_1^\varepsilon - \gamma_1^\varepsilon \partial_x^\varepsilon \hat{u}^\varepsilon - \gamma_2^\varepsilon \partial_x^\varepsilon \hat{v}^\varepsilon = 2\phi \left\{ \partial_y^\varepsilon (\cos \varphi^\varepsilon \hat{u}^\varepsilon) + \sin \varphi^\varepsilon \gamma_2^\varepsilon \right\} \] (106)

\[ \partial_t^\varepsilon \gamma_2^\varepsilon + \hat{u}^\varepsilon \partial_x^\varepsilon \gamma_2^\varepsilon + \hat{v}^\varepsilon \partial_y^\varepsilon \gamma_2^\varepsilon - \gamma_1^\varepsilon \partial_y^\varepsilon \hat{u}^\varepsilon - \gamma_2^\varepsilon \partial_y^\varepsilon \hat{v}^\varepsilon = -2\phi \left\{ \frac{\gamma_1^\varepsilon \sin \varphi^\varepsilon - \cos \varphi^\varepsilon \partial_y^\varepsilon \hat{v}^\varepsilon}{\gamma_1^\varepsilon} \right\} \] (107)

\[ u^\varepsilon = \hat{u}^\varepsilon + (z^\varepsilon - H^\varepsilon)\gamma_2^\varepsilon, \quad v^\varepsilon = \hat{v}^\varepsilon - (z^\varepsilon - H^\varepsilon)\gamma_1^\varepsilon \] (108)

\[ p^\varepsilon = p_0^\varepsilon + (z^\varepsilon - s^\varepsilon) (2\phi \cos \varphi^\varepsilon u^\varepsilon - g) \] (109)

\[ w^\varepsilon = u^\varepsilon \partial_x^\varepsilon H^\varepsilon + v^\varepsilon \partial_y^\varepsilon H^\varepsilon - (z^\varepsilon - H^\varepsilon) \left[ \partial_x^\varepsilon u^\varepsilon + \partial_y^\varepsilon v^\varepsilon + \gamma_2^\varepsilon \partial_x^\varepsilon H^\varepsilon - \gamma_1^\varepsilon \partial_y^\varepsilon H^\varepsilon \right] \]
\[ - \frac{1}{2} (z^\varepsilon - H^\varepsilon)^2 \left[ \partial_y^\varepsilon \gamma_1^\varepsilon - \gamma_2^\varepsilon \partial_y^\varepsilon \right] \] (110)

\((\hat{u}^\varepsilon, \hat{v}^\varepsilon) \text{ and } \gamma_i^\varepsilon \quad (i = 1, 2) \text{ do not depend on } z^\varepsilon\).

Notice that the average velocity is

\[ \bar{u}^\varepsilon = \frac{1}{h^\varepsilon} \int_{H^\varepsilon} u^\varepsilon dz^\varepsilon, \quad \bar{v}^\varepsilon = \frac{1}{h^\varepsilon} \int_{H^\varepsilon} v^\varepsilon dz^\varepsilon, \]

and then

\[ \bar{u}^\varepsilon = \hat{u}^\varepsilon + \frac{h^\varepsilon}{2} \gamma_2^\varepsilon, \quad \bar{v}^\varepsilon = \hat{v}^\varepsilon - \frac{h^\varepsilon}{2} \gamma_1^\varepsilon \]

Acknowledgements

This work has been partially supported by PGIDIT03PXIA11601PR project of Xunta de Galicia
REFERENCES


