MIXED FINITE ELEMENTS FOR NON-LINEAR MATERIAL PROBLEMS

R. Grimaldi, D. Addessi and V. Ciampi

Department of Structural Engineering
Via Eudossiana 18, 00184, University of Rome "La Sapienza", Italy
e-mail: roberta.grimaldi@uniroma1.it

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Abstract. The capability of mixed and enhanced finite elements to represent strain concentration in non-linear material plane stress problems is investigated. The classical 2 field and 3 field methods and an enhanced strain method are used, improved by the development of specific solution procedures to solve the non-linear problem and by the selection of ad hoc shape functions. All these methods, in particular mixed models, perform better than displacement-based methods and appear able to represent high strain concentrations, even when relatively coarse meshes are used.
1 INTRODUCTION

Mixed finite elements have been formulated to obtain a more direct approximation of
the stress state with respect to the approximations derived by the displacement-based
finite element methods.

They have been widely used for their capability to limit locking phenomena that may
arise when displacement-based finite elements are used to model problems such as plane
beams, incompressible materials and many others.

In this context, the most successful element is the Pian 4 node, assumed stress, element,
[1], for which the element stiffness matrix is derived from complementary energy and then
used in the framework of a displacement-based finite element code.

Mixed methods have been also used in the context of elasto-plastic material behaviour,
by satisfying the incremental consistency condition in integral form, [2] and [3].

Recently, [4], [5], [6], mixed methods have been proposed to model beam elements,
by taking advantage of the knowledge of the exact shape of the equilibrated shear and
moment fields. This makes possible to obtain, through the stress-strain law, the exact
distribution for the generalized strains, which can be much more complex than the one
derived directly from the displacements via the strain-displacement compatibility condi-
tions. If the material behaviour is non-linear and shows a softening branch, these models
enable a representation of localized strains, with no need either of enrichment of the
displacement shape functions, or of mesh refinements.

The very good performances of these beam elements are obviously due to the known
form of the equilibrated generalized stress fields, for every boundary (nodal) condition.
Therefore, it is always possible to adopt stress shape functions which include the exact
solution.

An extension of these methods to plane stress/strain 2-D models suffers from the lack
of an exact equilibrium solution, in terms of stresses, valid for any boundary condition.

In spite of these difficulties, it can be expected that, if a rich enough expansion is
assumed for the stress field, a good approximation can be obtained, in terms of stresses,
for a wide range of plane problems, even for coarse mesh discretizations. In this case, the
same advantages observed in the beam mixed models should be found, when a non-linear
material behaviour is assumed.

This paper, therefore, is devoted to test the capability of 2 and 3 field mixed formul-
ations to approximate the exact solution of plane non-linear material problems, starting
with coarse mesh discretizations.

Three models are evaluated: first the classical Hellinger-Reissner 2 field formulation,
with an ad hoc algorithm to solve the non-linear incremental stress-strain law; second
a Hu-Washizu 3 field formulation, with a specific choice of the independent strain shape
functions; third a modified version of the Simo-Rifai, [7], enhanced strain method, with the
improvement of localized strain shape functions. The latter model differs from many other
embedded-discontinuity enhanced strain methods, [8], for the reason that the localized
strains are still continuous over the element and become automatically active only when required by the material behaviour, with no need to include a bifurcation criteria, as usually done.

The paper is organized in three sections. In the first one, the formulation of the three elements is developed, with reference to the shape functions adopted for the independent fields. In the second, the algorithms used for the different models in a Newton-Raphson procedure are described. In the third, the performances of the models are evaluated, through the numerical analysis of some classical problems, in the presence of non-linear material behaviour, and compared with the ones obtained by using displacement-based formulations.

2 ELEMENT FORMULATION

In the following, three different formulations are presented, based on the assumptions of geometrically linear behaviour and plane stress state. The Hellinger-Reissner formulation is considered first, which assumes interpolation functions for the displacement and stress fields and is, therefore, a 2 field mixed method ($M_2$). On the other hand, based on the Hu-Washizu variational principle, a 3 field mixed method ($M_3$) is derived, where also the total strain field is interpolated. Finally, the enhanced strain ($ES$) formulation is also explored, which is based on the enrichment of the strain field description.

The displacement field is required to satisfy $C^0$ continuity, while the stress and strain fields must satisfy only $C^{-1}$ continuity requirement over the body domain.

In the $M_2$ approach, the element displacement $u(x)$, stress $\sigma(x)$ and compatible strain $\varepsilon_u(x)$ are expressed as:

$$u(x) = N_u(x) u_e \quad (1)$$
$$\varepsilon_u(x) = b_u(x) u_e = \mathcal{D} N_u(x) u_e \quad (2)$$
$$\sigma(x) = b_s(x) s_e \quad (3)$$

where $u_e$ and $s_e$ are the vectors of the nodal displacements and stresses, respectively; $N_u(x)$ and $b_s(x)$ are the corresponding interpolation matrices, containing the displacement and stress shape functions; $\mathcal{D}$ is the differential operator matrix. For the $M_3$ model, an independent interpolation for the strain field is used:

$$\varepsilon(x) = b_e(x) e_e \quad (4)$$

where $e_e$ contains the nodal strain parameters and $b_e(x)$ the interpolation functions for the strain field. Similarly, an enhancement of the strain field is used in the $ES$ model, by adopting an interpolation for the total strain in the form:

$$\varepsilon(x) = b_u(x) u_e + b_a(x) a_e \quad (5)$$
where \( a \) are the enhanced strain parameters and \( b_a(x) \) is the matrix containing the enhanced shape functions. The stress and strain parameters \( s_e \) and \( e_e(a_e \text{ for the ES model}) \) are defined as local variables, at the element level, and are not involved in the global solution procedure. They may be defined as generalized stress and strain states in the element.

The same interpolation functions for the stress field is assumed in the three models, while the strain field is interpolated differently in the M3 and ES models.

In the following, 4 node and 8 node quadrilateral elements are developed for all the models, assuming always a bilinear transformation \( x = \varphi(\xi) \) from the parent element domain \( \xi = (\xi, \eta) \) to the actual one \( x = (x, y) \). The shape functions are all defined in the parent coordinate system \( \xi \), giving origin to an isoparametric (4 node) and hypoparametric (8 node) element, respectively. For the stress and strain fields the usual push-pull transformation between the parent element and the actual one is used, in order to satisfy the invariance of the solution in terms of stress and strain under a rotation of the global coordinate system, [7], [9]. Therefore, if \( \Sigma(\xi) \) and \( E(\xi) \) are the stress and strain vectors in the parent element, the real stress and strain vectors in the actual element will be:

\[
\sigma(x(\xi)) = F_0 \Sigma(\xi) \quad \varepsilon(x(\xi)) = \frac{\det J_0}{\det J(\xi)} F_0^T E(\xi) \tag{6}
\]

where \( J(\xi) \) is the gradient of the transformation and \( J_0 = J(\xi = 0) \). \( F_0 \) is a matrix whose components are:

\[
F_0 = \begin{bmatrix}
J_{21}^2 & J_{21} J_{12} & 2J_{11} J_{12} \\
J_{12} J_{21} & J_{22}^2 & 2J_{21} J_{22} \\
J_{11} J_{21} & J_{12} J_{22} & (J_{11} J_{22} + J_{12} J_{21})
\end{bmatrix}_{\xi=0} \tag{7}
\]

with \( J_{\alpha\beta} = \partial\varphi_\alpha / \partial\xi_\beta \). This kind of transformation allows to satisfy the equivalence of the internal work between the parent and actual configuration.

### 2.1 Two Field Mixed Model

Many suggestions can be found in the literature in order to select a suitable interpolation for the stress field in a quadrilateral element with a given displacement field. It is well known that, when a Hellinger-Reissner type mixed formulation is adopted, the existence and uniqueness of the displacement solution require the following condition to be satisfied, [10]:

\[
n_s \geq n_d = n_u - n_r \tag{8}
\]

where \( n_s \) is the number of the element stress parameters and \( n_d \) the number of the element deformation modes, equal to the difference between \( n_u \) (the total number of displacement parameters) and \( n_r = 3 \) (the number of the element plane rigid motions). Moreover, the stress shape functions, collected in \( b_s \), should not contain any element orthogonal to the compatible strain shape functions collected in \( b_u \), [11].
These expansions, however, may still contain terms that allow the element to be self-equilibrated, i.e. hyperstatic terms.

Pian and Sumihara, [12], and later Bilotta and Casciaro, [13], suggested a method to select a reduced number of stress parameters, among the coefficients of a complete polynomial expansion, based on an equilibrium condition in integral form.

Here it is simply required that, after applying the push-pull transformation (6) to the stress components from the parent element to the actual one, no self-equilibrated stress distributions be allowed in the element, i.e. that:

$$u_e^T \int_{B_e} b_a^T b_s \, dB \, s_e = 0 \quad \forall \quad u_e \iff s_e = 0$$

(9)

So, in the case of the 4 node element, the same 5 parameters as for the widely used Pian-Sumihara element are obtained for $s_e$, while, for the 8 node element, a 13 parameters $s_e$ vector is derived, which differs from the 14 parameters vector proposed by Bilotta and Casciaro, in particular for the expansion assumed for the term $\sigma_{12}$.

Indeed the matrix $b_s$, for the 4 node element, assumes the well known form:

$$b_s(\xi) = \begin{bmatrix}
1 & 0 & 0 & a_3^2 \xi & a_1^2 \eta \\
0 & 1 & 0 & b_3^2 \xi & b_1^2 \eta \\
0 & 0 & 1 & a_3 b_3 \xi & a_1 b_1 \eta^2 
\end{bmatrix}$$

(10)

while, for the 8 node element:

$$b_s(\xi) = \begin{bmatrix}
1 & 0 & 0 & \xi & \eta & 0 & 0 & 0 & 0 & a_3^2 \xi^2 & \xi \eta & 0 & a_1^2 \eta^2 \\
0 & 1 & 0 & 0 & \xi & \eta & 0 & 0 & b_3^2 \xi^2 & 0 & \xi \eta & b_1^2 \eta^2 \\
0 & 0 & 1 & 0 & 0 & 0 & \xi & \eta & a_3 b_3 \xi^2 & 0 & 0 & a_1 b_1 \eta^2 
\end{bmatrix}$$

(11)

where $a_1$, $a_3$, $b_1$, $b_3$ are the components of the transformation gradient $J_0$ and are expressed as a function of the corner-node coordinates:

$$a_1 = \frac{1}{4}(-x_1 + x_2 + x_3 - x_4); \quad a_3 = \frac{1}{4}(-x_1 - x_2 + x_3 + x_4)$$

$$b_1 = \frac{1}{4}(-y_1 + y_2 + y_3 - y_4); \quad b_3 = \frac{1}{4}(-y_1 - y_2 + y_3 + y_4)$$

2.2 Three Field Mixed Model

For the three field mixed model, since the strain nodal parameters have to be condensed at the element level, the only requirement for the existence and uniqueness of the solution in terms of stress, for each element, is:

$$n_e \geq n_s$$

(12)

where $n_e$ is the number of the element deformation parameters. The strain interpolation functions are selected so to be able to represent the non smooth distributions that may be expected, in presence of a strongly non-linear material behaviour.
A suitable choice is represented by Lagrange polynomials, $L_{g_i}(\xi)$, (which take value 1 at the integration point $i$ and 0 at all the others), whose number is equal to that of the Gauss integration points in the element. Each of them is equivalent to a unity step function, extended to an area equal to the weight associated with the Gauss point $i$. Then, the strain components in the parent element can be expressed as:

$$
\begin{bmatrix}
E_{11} & E_{12} & \cdots & E_{1m} \\
E_{21} & E_{22} & \cdots & E_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
E_{m1} & E_{m2} & \cdots & E_{mm}
\end{bmatrix}
= \begin{bmatrix}
L_{g_1}(\xi) & 0 & \cdots & 0 \\
0 & L_{g_1}(\xi) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & L_{g_n}(\xi)
\end{bmatrix}
\begin{bmatrix}
\epsilon_{e1} \\
\epsilon_{e2} \\
\vdots \\
\epsilon_{em}
\end{bmatrix}
$$

(13)

After the push-pull transformation (6) is applied to the strain components, the strain shape function matrix $b_e(\xi)$ is easily obtained.

In the following, the number of Gauss points used is $n = 5 \times 5 = 25$ and the total number of strain degrees of freedom is $m = 3n = 75$, so that the condition (12) is verified and a rich enough representation of the strain field is allowed.

The above selected expansion for the strain field allows to obtain a simplified form of the element tangent stiffness matrix, expressed as the collection of the $3 \times 3$ tangent stiffness matrices evaluated at the Gauss points, making computationally more efficient the inversion operations required during the condensation process at the element level.

Moreover, in the linear elastic case, the chosen interpolation for the strain field satisfies the equivalence conditions, [11], between the $M3$ and the $M2$ models.

### 2.3 Enhanced Strain Model

The enhanced strain formulation is based on the following additive decomposition of the total strain $\varepsilon$, [7]:

$$
\varepsilon = \varepsilon_u + \varepsilon_a
$$

(14)

where $\varepsilon_u$ is the compatible part of the strain, which satisfies pointwise the strain-displacement compatibility conditions, while $\varepsilon_a$ is the enhanced part of the strain, which is required to satisfy the compatibility conditions only in integral form. In particular, by enforcing the stationarity of the Hu-Washizu functional, the compatibility condition for the enhanced part of the strain is derived as:

$$
\int_{B_e} \sigma^T \varepsilon_a dB = 0
$$

(15)

It follows that the enhanced strains are self-strains, orthogonal to every assumed stress state in $B_e$.

Here, it is assumed that the enhanced part of the total strain can be further decomposed as:

$$
\varepsilon_a = \varepsilon_a^s + \varepsilon_a^h
$$

(16)
In this case $\varepsilon_a^s$ is the standard part of the enhanced strain, which for plane models is usually introduced to limit locking, and whose components are well known for the 4 node plane element, [7], while $\varepsilon_a^h$ is the higher order part of the total strain, which is activated only in presence of strain concentration inside the element due to non-linear material behaviour. To ensure that the last component, $\varepsilon_a^h$, does not affect the strain distribution during a linear elastic load path, the shape functions used for $\varepsilon_a^h$ are required to be orthogonal, in the sense of the strain energy, to the shape functions assumed for $\varepsilon_a^s$ and $\varepsilon_u$. Therefore, if the expansions for the strain are given by the expression (2) for the compatible part and, for the enhanced part, by:

$$
\varepsilon_a = \varepsilon_a^s + \varepsilon_a^h = \begin{vmatrix}
\mathbf{b}_a^s & \mathbf{b}_a^h
\end{vmatrix}
\begin{pmatrix}
\mathbf{a}_a^s \\
\mathbf{a}_a^h
\end{pmatrix}
$$

(17)

it is required that, whenever the material constitutive matrix $C(\xi)$ is constant in the element, there is no coupling in the energy sense between $\varepsilon_a^h$ and $\varepsilon_a^s$ or $\varepsilon_u$. Indeed, it must be:

$$
\int_{B_e} \varepsilon_a^h^T C \varepsilon_u dB = 0 \quad \forall \varepsilon_u
$$

(18)

and:

$$
\int_{B_e} \varepsilon_a^h^T C \varepsilon_a^s dB = 0 \quad \forall \varepsilon_a^s
$$

(19)

For the 4 and 8 node elements the shape function matrices, $\mathbf{b}_a^s$, are, respectively:

$$
\mathbf{b}_a^s = \frac{1}{|\det J(\xi)|} \begin{pmatrix}
\frac{\partial \xi}{\partial \xi} & \frac{\partial \eta}{\partial \xi} & \frac{\partial \zeta}{\partial \xi} \\
\frac{\partial \xi}{\partial \eta} & \frac{\partial \eta}{\partial \eta} & \frac{\partial \zeta}{\partial \eta} \\
-2a_3 b_3 & -2a_1 b_1 & c \zeta
\end{pmatrix}
$$

(20)

$$
\mathbf{b}_a^h = \frac{1}{|\det J(\xi)|} \begin{pmatrix}
\frac{\partial f(\xi)}{\partial \xi} & \frac{\partial f(\eta)}{\partial \xi} & \frac{\partial f(\zeta)}{\partial \xi} \\
\frac{\partial f(\xi)}{\partial \eta} & \frac{\partial f(\eta)}{\partial \eta} & \frac{\partial f(\zeta)}{\partial \eta} \\
-2a_3 b_3 f(\xi) & -2a_1 b_1 f(\eta) & c \eta
\end{pmatrix}
$$

(21)

where $c = (b_1 a_3 + a_1 b_3)$, $d = a_1 a_3$, $e = b_1 b_3$ and $f(\zeta) = \frac{1}{2}(3\zeta^2 - 1)$.

It could be easily verified that, assuming the same interpolation functions for the stress field in the M2 and ES models, the expansion adopted for $\varepsilon_a^s$ satisfies the orthogonality condition (15). Moreover, it should be pointed out that, when the above assumptions hold, and when meshes are not distorted and a linear elastic model is used, no differences between the M2 and ES models are found. Indeed, the expansion adopted for the enhanced strain $\varepsilon_a^s$ also satisfies the equivalence condition, [14], between the ES and M2 methods.

For both the 4 and the 8 node elements, a suitable choice for the expansion of $\varepsilon_a^h$ is based on the Legendre polynomials of higher order in $\xi$. Explicitly, the components of $\varepsilon_a^h$
in the parent element can be expressed as:

\[
\begin{bmatrix}
E_{\alpha 11}^h \\
E_{\beta 22}^h \\
2E_{\alpha 12}^h
\end{bmatrix}
= |\text{det} J(\xi)| \begin{bmatrix}
P_{n1}(\xi) & 0 & 0 & P_{n1}(\eta) & \cdots & 0 \\
0 & P_{n1}(\xi) & 0 & 0 & \cdots & 0 \\
0 & 0 & P_{n1}(\xi) & 0 & \cdots & P_{n2}(\eta)
\end{bmatrix}
\begin{bmatrix}
a_1^h \\
a_2^h \\
a_3^h \\
a_4^h \\
\vdots \\
a_m^h
\end{bmatrix}
\]  

(22)

where \(P_{n1}\) is the Legendre polynomial of \(n_1\) degree (minimum), and \(P_{n2}\) is the Legendre polynomial of \(n_2\) degree (maximum). It has to be noted that, if the values \(n_1 = 3\) and \(n_1 = 4\) for the lower degree polynomial are assumed, respectively, for the 4 node and the 8 node element, the orthogonality conditions (15) are automatically satisfied together with (18, 19), these last two only in the linear elastic range. In the following, the value \(n_2 = n_1 + 1\) is used, so that the total number of the enhanced parameters used for the expansion of \(\epsilon^h\) is \(m = 12\).

### 2.4 Constitutive models

Three different constitutive laws are considered in the applications which follow. Firstly, a linear elastic behaviour is used. The second model is based on a non-linear elastic relationship defined as follows:

\[
\sigma = C\varepsilon = \frac{E(\varepsilon)}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1 - \nu}{2}
\end{bmatrix} \varepsilon, \quad E(\varepsilon) = \frac{E_0}{1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2}}
\]

(23)

where \(E_0\) and \(\nu\) are the Young’s modulus and Poisson’s ratio, respectively, while \(E(\varepsilon)\) is the non-linear elastic modulus, whose variation is governed by an equivalent measure of the principal strains \(\varepsilon_1\) and \(\varepsilon_2\).

Finally, a strain-softening constitutive model is adopted, based on the introduction in the stress-strain law of a scalar damage variable \(d\), limited in the range \([0, 1]\), and subjected to the thermodynamical irreversibility condition:

\[
\sigma = (1 - d)C_0\varepsilon \quad \text{with} \quad 0 \leq d \leq 1, \quad d \geq 0
\]

(24)

where \(C_0\) is the plane stress elastic matrix. An exponential evolution law is considered for \(d\), expressed by:

\[
d = 0 \quad \text{if} \quad \varepsilon_{eq} < \varepsilon_0
\]

\[
d = 1 - \frac{\varepsilon_{eq}}{t_{eq}\varepsilon_0^{\varepsilon_{eq}}/k} \quad \text{if} \quad \varepsilon_{eq} > \varepsilon_0
\]

where the damage associated variable, governing its evolution, is defined as:

\[
\varepsilon_{eq} = \sqrt{\sum_i (\varepsilon_i)^2} \quad i = 1, 2
\]
and $\varepsilon_0$ and $k$ are material parameters, governing the damage initial threshold and the rate of the damage evolution, respectively.

3 Computational Issues and Solution Algorithms

For each of the three models a generalized Newton-Raphson solution procedure, with condensation of the stress and strain independent variables at the element level, is used.

In the $M^2$ model, the set of non-linear equations to be solved in weak form is given by the equilibrium equations and by the compatibility strain-displacements equations, which can be expressed, for a finite element discretization, respectively as:

$$\sum_e \delta u_e^T (B_{us} s_e - F_{oe}) = \sum_e \delta u_e^T r_u (s_e) = 0 \quad (25)$$

$$\sum_e \delta s_e^T (B_{us} u_e - u_i (s_e)) = \sum_e \delta s_e^T r_s (u_e, s_e) = 0 \quad (26)$$

where:

$$B_{us} = \int_{B_e} b_i^T b_s dB \quad \text{and} \quad u_i (s_e) = \int_{B_e} b_i^T \varepsilon (\sigma) dB \quad (27)$$

The internal displacements $u_i (s_e)$ are the dual quantities of the internal forces in a displacement-based finite element model and represent the kinematic quantities, obtained from the stress state via the stress-strain law, which are associated to the stress parameters $s_e$ in the internal work expression. The matrix $B_{us}$ is the element equilibrium matrix, which transforms the stress variables $s_e$ in the static variables, dual with respect to the nodal displacements $u_e$.

Beaing the displacement parameters, $u$, defined as global variables, the equilibrium equations are imposed at the global level after the assembling process. On the contrary, being the stress parameters, $s_e$, defined as local element variables, the compatibility equations are imposed independently for each element. Therefore, during each load-step, the Newton-Raphson solution procedure, at global level, requires that:

$$\sum_e r_{ue}^{i+1} = \sum_e \left( r_u (s_e^i) + \frac{\partial r_u (s_e^i)}{\partial s_e} \Delta s_e^i \right) = 0 \quad (28)$$

while, for each element:

$$r_{se}^{i+1} = r_s (u_e^i, s_e^i) + \frac{\partial r_s (u_e^i, s_e^i)}{\partial u_e} \Delta u_e^i + \frac{\partial r_s (u_e^i, s_e^i)}{\partial s_e} \Delta s_e^i = 0 \quad (29)$$

where:

$$D_{ss}^i = \frac{\partial r_s (s_e^i)}{\partial s_e} \quad (30)$$
By expressing from (29) the stress degrees of freedom increment $\Delta s^i_e$ at the element level as a function of the displacement degrees of freedom increment $\Delta u^i_e$, the condensed equilibrium equations are obtained:

$$
\sum_e r^{i+1}_{ue} = \sum_e \left( r^i_{ue} + B_{us} \left( D_{ss}^i \right)^{-1} r^i_s + B_{us} \left( D_{ss}^i \right)^{-1} B_{us}^T \Delta u^i_e \right) = (31)
$$

$$
\sum_e \left( r^i_{ue} + K^i_e \Delta u^i_e \right) = 0
$$

where $K^i_e = B_{us} \left( D_{ss}^i \right)^{-1} B_{us}^T$ represents the condensed element stiffness matrix and $r^i_{ue} = r^i_{ue} + B_{us} \left( D_{ss}^i \right)^{-1} r^i_s$ represents the condensed element equilibrium residual, which accounts also for the residual on the compatibility equations. The element equilibrium equations can, then, be assembled and solved at the global level as in any other displacement-based finite element code. The iteration algorithm can be resumed as follows:

1. After the global set of equations (31) is solved, the displacement increment $\Delta u^i_e$ is determined for each element and the total displacement $u^i_e$ is updated:

$$
u^i+1_e = u^i_e + \Delta u^i_e \quad (32)
$$

2. The stress increment is determined by the (29) for each element and the stress parameters are updated:

$$
\Delta s^i_e = \left( D_{ss}^i \right)^{-1} B_{us}^T \Delta u^i_e + \left( D_{ss}^i \right)^{-1} r^i_s
$$

$$
s^{i+1}_e = s^i_e + \Delta s^i_e \quad (33)
$$

3. The element state is updated, evaluating the new element residuals and the new tangent compliance matrix as:

$$
r^{i+1}_s = B_{us}^T u^{i+1}_e - u_e \left( s^{i+1}_e \right)
$$

$$
r^{i+1}_{ue} = B_{us} s^{i+1}_e - F_e
$$

$$
D^{i+1}_{ss} = \frac{\partial u_e \left( s^{i+1}_e \right)}{\partial s_e}
$$

The residuals $r^{i+1}_s$, $r^{i+1}_{ue}$ and the compliance matrix $D^{i+1}_{ss}$ are now used to perform again the element condensation (31). The iteration cycle is repeated until a selected norm of the global condensed residuals $r_u$ is sufficiently small.

Whenever the stress-strain law is not available in the inverse form, for example when it exhibits a softening branch, a specific algorithm has to be developed to evaluate the internal displacement vector $u_e \left( s_e \right)$ and the element compliance matrix $D_{ss}$. Here we assume that, for a given stress level at the Gauss point at the current iteration $i$, $\sigma^i = b_s s^i_e$, 

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the corresponding strain can be derived by using the last iteration updated material stiffness matrix $C\left(\varepsilon^{i-1}\right)$:

$$
\varepsilon^i = (C\left(\varepsilon^{i-1}\right))^{-1} \sigma^i = C^{(i-1)^{-1}} \sigma^i \tag{35}
$$

The current material stiffness matrix can, then, be evaluated at each Gauss point as a function of the updated strain:

$$
C^i = C\left(\varepsilon^i\right) \tag{36}
$$

Finally, the element residual on the compatibility equation, $r^i_s$, and the secant element compliance, $D^{ss}_i$, at the iteration $i$ are given, respectively, by:

$$
\begin{align*}
    r^i_s &= B^T_{us} u^i_u - u^i_s \left(\sigma^i\right) = B^T_{us} u^i_u - \int_{B_e} b^T_{s} C^{i-1} \sigma dB \tag{37} \\
    D^{ss}_i &= \int_{B_e} b^T_{s} C^{i-1} b_s dB \tag{38}
\end{align*}
$$

A graphical representation of the path followed by the element variables during the modified iterative algorithm is given in Figure 1.

![Figure 1: Solution algorithm for the non-linear M2 model.](image)

In the M3 model, the set of non-linear equations to be solved in weak form is given by the equilibrium equations, the constitutive law and the compatibility strain-displacements equations. For a finite element discretization, the former is exactly the same as in the M2 model (25), while the latter two can be expressed as:

$$
\begin{align*}
    \sum_e \delta e^T_e \left( p^e_e (e_e) - B_{es} s_e \right) &= \sum_e \delta e^T_e r^e_e (e_e, s_e) = 0 \tag{39} \\
    \sum_e \delta s^T_e \left( B^T_{us} u^e_u - B^T_{es} e_e \right) &= \sum_e \delta s^T_e r^e_s (u^e_u, e_e) = 0 \tag{40}
\end{align*}
$$

where:

$$
B_{es} = \int_{B_e} b^T_{s} dB \quad \text{and} \quad p^e_e (e_e) = \int_{B_e} b^T_{e} \sigma (e_e) dB \tag{41}
$$
Now $\mathbf{B}_{es}$ is the matrix coupling the stress-strain parameters and $\mathbf{p}_t(\mathbf{e}_e)$ is the vector of the internal forces dual with respect to the element strain parameters. The solution strategy follows the same path resumed for the M2 method. The element state updating and condensation procedure is described in Figure 2.

![Figure 2: M3 and ES models - element state updating and condensation procedure.](image)

Finally, for the ES model the set of non-linear equations to be solved in weak form is given by the equilibrium equations and by the stress-strain law in weak form, which now are expressed as:

$$
\sum_{e} \delta \mathbf{u}_e^T \left( \mathbf{f}_t(\mathbf{u}_e, \mathbf{a}_e) - \mathbf{F}_e \right) = \sum_{e} \delta \mathbf{u}_e^T \mathbf{r}_{u}(\mathbf{u}_e, \mathbf{a}_e) = 0 \quad (42)
$$

$$
\sum_{e} \delta \mathbf{a}_e^T \left( \mathbf{p}_t(\mathbf{u}_e, \mathbf{a}_e) \right) = \sum_{e} \delta \mathbf{a}_e^T \mathbf{r}_a(\mathbf{u}_e, \mathbf{a}_e) = 0 \quad (43)
$$

where:

$$
\mathbf{f}_t(\mathbf{u}_e, \mathbf{a}_e) = \int_{B_e} \mathbf{b}_u^T \mathbf{\sigma}(\mathbf{u}_e, \mathbf{a}_e) \, d\mathbf{B}
$$

$$
\mathbf{p}_t(\mathbf{u}_e, \mathbf{a}_e) = \int_{B_e} \mathbf{b}_u^T \mathbf{\sigma}(\mathbf{u}_e, \mathbf{a}_e) \, d\mathbf{B}
$$

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Now \( \mathbf{f}_i(\mathbf{u}_e, \mathbf{a}_e) \) are the internal forces dual to the nodal displacements \( \mathbf{u}_e \), while \( \mathbf{p}_i(\mathbf{u}_e, \mathbf{a}_e) \) are the internal forces dual with respect to the enhanced strain parameters \( \mathbf{a}_e \). Again, the solution strategy follows exactly the same path resumed for the M2 method and the element state updating and condensation procedure is described in Figure 2.

4 NUMERICAL APPLICATIONS

In the following, some numerical 2D tests are presented. The different elements introduced before, based on the M2, M3 and ES formulations, respectively, are used throughout. The results obtained by using a standard displacement-based approach \((D)\) are also shown for comparison. In all the applications, and for all the models, the number of Gauss points used for the numerical integration over an element is 25. All the models are implemented by using the finite element analysis program FEAP 7.4, [9].

4.1 Linear elastic cantilever beam

The first application tests the performances of the proposed models for highly distorted meshes, using a linear elastic constitutive law. A cantilever beam, (Fig. 3), is subjected in the first analysis (Test 1) to a pure bending load and in the second test (Test 2) to a shear load applied at the free end.

Figure 3: Cantilever beam - geometry and boundary conditions.

Figure 4: Test 1 and Test 2 - nondimensional displacement at the free end versus distorsion \( a \).
The geometrical and mechanical nondimensional parameters are assumed as follows: 
\( L = 10, \ H = 2; \ E_0 = 1500, \ \nu = 0.25 \). The applied forces are \( F = 1000 \) for the Test 1 and \( F = 150 \) for the Test 2. A two element mesh is used and the parameter controlling the mesh distortion is indicated as \( a \). In Figure 4, the normalized vertical displacement \( \frac{v_A}{v_{\text{exact}}} \) versus the nondimensional mesh distortion parameter \( \frac{a}{L} \) is reported, where \( v_A \) is the computed displacement value at the free end and \( v_{\text{exact}} \) is the corresponding analytical exact solution; both 4 node and 8 node elements are used. As it can be seen, the \( M_2, M_3 \) and \( ES \) models show always better performances than the \( D \) model for both the 4 node and 8 node cases. As expected, the solution obtained with the \( M_2 \) and \( M_3 \) models coincide, while the performance of the \( ES \) model gets worse as the mesh distortion parameter \( a \) increases.

### 4.2 Non-linear elastic slab with a hole

In the second application, the structural behaviour of a square slab with a square hole subjected to a given vertical displacement \( u \) (Fig. 5) is analyzed. In this case, the non-linear elastic stress-strain law (23) is adopted. Due to the double symmetry, only a quarter of the slab is analyzed. The length of the slab side and of the hole side are \( L = 0.1 \ m \) and \( l = L/10 \), respectively. The initial Young’s modulus and the Poisson’s ratio are \( E_0 = 210000000 \ K N/m^2 \) and \( \nu = 0.3 \). The four different meshes, shown in Figure 6, characterized by an increasing number of elements are compared.

![Figure 5: Slab with a hole - geometry and boundary conditions.](image)

![Figure 6: Slab with a hole - MESH1 (8 EL), MESH2 (116 EL), MESH3 (448 EL), MESH4 (1760EL).](image)

Figure 7 depicts the distribution of the strain component \( \varepsilon_{22} \) along \( x \) (for \( y = l \)) for the
four meshes used. It is observed that the highest values of the normal strain at the hole corner are obtained with the 8 node elements, based on the $M_2$ and $M_3$ formulations, which are able to better describe the strain localization.

In Figures 8 A) and 8 B) the nondimensional load, $P/E_0$, evaluated at the top loaded face, versus the number of elements is reported, for a fixed value of the given displacement, $u/L = 20$. It can be seen that the convergence to the exact solution is significantly faster, in comparison with the displacement ($D$) model, when the mixed ($M_2$ and $M_3$) and the
enhanced strain model (ES) are used, in particular for the 4 node elements. Figures 8 C) and 8 D) show the computing time versus the number of elements: while the $M_3$ model appears to be the most time consuming, the $M_2$ model is almost equivalent to the displacement-based ($D$) model in terms of computational effort.

![Figure 8: Slab with a hole - nondimensional load at the top face (A and B) and computing time (C and D) versus number of elements.](image)

4.3 Three-points bending test on a beam with strain-softening behaviour

The last application presented is a three-points bending test performed on a beam with a strain-softening constitutive behaviour based on the exponential damage evolution law (24). Because of simmetry, only one half of the beam is analyzed (Fig. 9). The beam length and thickness are $L = 0.1\, m$ and $H = 0.02\, m$, respectively. The initial Young’s modulus and Poisson’s ratio are the same as in the previous test on the slab with a hole. Two different meshes are used, with 36 (mesh1) and 136 (mesh2) elements respectively (Fig. 10).

![Figure 9: Three-points bending test - beam geometry and boundary conditions.](image)
The test is performed by using only the $M2$ and the $D$ models. In Figure 11 the global force-displacement curves are presented for the two meshes considered. It can be seen that the $M2$ model shows a steeper softening branch than the $D$ model, due to the better capability to capture the strain localization near the crack tip, as also shown by the strain distribution presented in Figure 12, at different levels of the given displacement. The damage distribution at different displacement levels is, finally, depicted in Figures 13 and 14, for the 4 node and the 8 node $M2$ element, respectively; here, again, the concentration of the damaging process around the middle section and the different behaviour of the 4 node and the 8 node elements in capturing the localization can be clearly noted.

![Figure 10: Three-points bending test - beam discretizations.](image1)

![Figure 11: Three-points bending test - global load-displacement curves.](image2)

![Figure 12: Three-points bending test - strain distribution along $x$.](image3)
Obviously, the numerical results obtained for this last test are affected by the expected mesh-dependency in the softening range, connected with the use of a strain-softening constitutive law. Then, the adoption of a regularization technique would be necessary in order to reach more objective and comparable solutions.

The interest, in this case, is only addressed to show the relative capabilities of the different models to capture the strain localization.

Figure 13: Three-points bending test - damage distribution for the 4 node $M_2$ elements.

Figure 14: Three-points bending test - damage distribution for the 8 node $M_2$ elements.
5 CONCLUSIONS

After verifying the better behaviour of the mixed formulations in the linear elastic case, both in terms of mesh distortion sensitivity and in terms of locking free behaviour, the performances of three mixed finite elements models have been tested by analyzing two non-linear material problems, one with non-linear elastic behaviour and the other with damage-induced softening branches.

The first non-linear material test shows that all the models perform better than the displacement-based one, both in terms of evaluation of the overall structural stiffness and in terms of local representation of the strains. In particular, the 4 node displacement-based element seems to be too stiff, not only for problems which suffer from locking behaviour, as the cantilever elastic beam problem, but also for more general structural problems. For the 8 node element, the differences between the mixed models and the displacement-based model are more evident, when an accurate evaluation of the local strain values is required. For what concerns the differences between the three mixed models, it can be observed that, due also to the interpolations assumed for the strain field, the $M_3$ model is less efficient from a computational point of view, while the enhanced strain model is less performant with respect to the $M_2$ and $M_3$ field models, in particular for coarse meshes.

For these reasons, only the behaviour of the $M_2$ model has been examined in the second non-linear test, with a damaging material. From this test it appears that, when the softening branch along the load-displacement curve is reached, the differences between the 2 field model and the displacement-based model become more and more prominent. The test has been performed using two meshes that are relatively coarse for the considered problem; the $M_2$ model shows always a higher capability to represent the strain localization behaviour, especially when 8 node elements are used.

In general it can be concluded that, whenever a stress-based model is able to represent with good approximation the exact stress field, it gives also a good representation of the strain field for every material behaviour. It always performs better than a displacement-based model, as long as a strongly irregular strain field is expected.
REFERENCES


