THE DOMINANT WAVE-CAPTURING FINITE-VOLUME
SCHEME FOR SYSTEMS OF HYPERBOLIC
CONSERVATION LAWS

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Abstract. More robust developments of schemes for hyperbolic systems, that avoid dependence upon a characteristic decomposition have been achieved by employing some form of a Lax-Friedrichs (LF) based flux. Such schemes permit the construction of higher order approximations without recourse to characteristic decomposition. This is achieved by using the maximum eigenvalue of the hyperbolic system within the definition of the LF flux. The current literature on these schemes only appears to indicate success in this regard, with no investigation of the effect of the additional numerical diffusion that is inherent in such formulations.

In this paper the foundation for a new scheme is proposed which relies on the detection of the dominant wave in the system. This scheme is designed to permit the construction of lower and higher order approximations without recourse to characteristic decomposition while avoiding the excessive numerical diffusion that is inherent, even within the least dissipative (LF) flux.
1 INTRODUCTION

Many locally conservative schemes have been developed for systems of hyperbolic conservation laws [1]. The most successful high resolution schemes, in terms of actual front resolution, depend upon a characteristic decomposition of the system. The decomposition leads to optimal upwind schemes where upwind directions can be resolved according to the characteristic wave components and upwind approximations applied with minimum dissipation. Roe’s approximate Riemann solver [2] is one of the most popular schemes of this type. The method has excellent shock capturing capabilities and the relative simplicity of the method makes it one of the most efficient decomposition schemes when compared to rival formulations.

However, the discovery of a scheme that can provide front resolution that is comparable with that of the best upwind schemes, while avoiding the need for characteristic decomposition continues to present a major challenge in this area, e.g. [3].

The standard approach to this problem, particularly for steady state problems, involves employing schemes that are essentially ”central difference” in character together with some form of artificial dissipation.

More robust developments of schemes for systems, that avoid dependence upon a characteristic decomposition have been achieved by employing some form of a Lax-Friedrichs (LF) related flux [4], [3] and later in [5] and [6]. Such schemes permit the construction of higher order approximations without recourse to characteristic decomposition. This is achieved by using the maximum eigenvalue of the hyperbolic system within the definition of the LF flux. The recent literature on these schemes only appears to indicate success in this regard, with no investigation of the effect of the additional numerical diffusion that is inherent in such formulations.

In this paper the foundation for a new scheme is proposed which relies on the detection of the dominant wave in the system [7]. This scheme is designed to permit the construction of lower and higher order approximations without recourse to characteristic decomposition while avoiding the excessive numerical diffusion that is inherent, even within the least dissipative (LF) flux (i.e. the Local Lax-Friedrichs (LLF) flux approximation).

This paper describes the motivation and derivation of the dominant wave capturing technique. The formulation ensures that local conservation is maintained and is developed within a general finite volume framework. Low order and higher order versions of the method are presented on structured and unstructured grids.

The application includes the Euler equations of compressible flow. Comparisons between the new method and the LLF schemes reveal clear weaknesses of the LLF based formulations. Some classical flow problems are presented where the LLF schemes fail to detect discontinuities that are known to be present in the physical solution. In contrast, the new dominant wave scheme is able to capture the discontinuities while using exactly the same grids and equivalent levels of accuracy in terms of polynomial approximation. The results presented demonstrate the benefits of the dominant wave formulation, for
both low order and higher order approximations on structured and unstructured grids.

Finally it is noted that the dominant wave formulation offers the benefits of being directly applicable to other systems of hyperbolic conservation laws without requiring a characteristic decomposition.

2 FLOW EQUATIONS

The schemes presented here are applicable to hyperbolic systems of the form

\[
\int_\Omega \frac{\partial u}{\partial t} dV + \int_\Omega \frac{\partial F(u)}{\partial x} + \frac{\partial G(u)}{\partial y} dV = 0
\]  

where the integral is over volume \( \Omega \). The Euler equations of compressible flow are considered in this paper with

\[
\begin{align*}
\mathbf{u} &= (\rho, \rho u, \rho v, E)^T \\
\mathbf{F}(\mathbf{u}) &= (\rho u, \rho u^2 + p, \rho uv, u(E + p))^T \\
\mathbf{G}(\mathbf{u}) &= (\rho v, \rho uv, \rho v^2 + p, v(E + p))^T
\end{align*}
\]  

(2)

Here \( \rho \), \( p \) and \( E \) are the density, pressure and energy per unit volume of an ideal gas \((u, v)\) the Cartesian components of velocity and

\[
E = \rho \left[ \frac{1}{2} q^2 + p/\rho(\gamma - 1) \right]
\]

(3)

\( \gamma \) being the ratio of specific heat capacities, \( q^2 = u^2 + v^2 \) and sound speed \( a = \sqrt{\gamma p/\rho} \).

2.1 Boundary conditions

For the initial value problem (IVP) field data is prescribed. For initial boundary value problems (IBVP), considered here in two-dimensions, an initial flow field is prescribed together with boundary values which are assigned according to the number of inward pointing characteristics, e.g. \[1\] for further discussion. Zero normal flow is imposed on solid walls. The zero-normal flow condition is imposed weakly via the flux, on a solid wall the Euler flux reduces to being a function of pressure.

3 Approximate Riemann Solvers with Upwinding

First we shall consider schemes in one spatial dimension on a computational grid with discrete nodes \( x_i = i\Delta x \) and time \( t^n = n\Delta t \). Before moving to systems we briefly recall that for a scalar equation of the form

\[
\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0
\]

(4)
the standard first order upwind scheme can be written as

$$ \begin{align*} u_i^{n+1} &= u_i^n - \frac{\Delta t}{\Delta x} (f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n) \tag{5} \end{align*} $$

where the approximate flux is defined by

$$ f_{i+\frac{1}{2}}^n = \frac{1}{2} (F(u_i^n) + F(u_{i+1}^n) - |\lambda_{i+\frac{1}{2}}| (u_{i+1}^n - u_i^n)) \tag{6} $$

and

$$ \lambda_{i+\frac{1}{2}} = \left( \frac{(F(u_{i+1}^n) - F(u_i^n))}{(u_{i+1}^n - u_i^n)} \right) |(u_{i+1}^n - u_i^n)| > \epsilon \quad \frac{(u_{i+1}^n - u_i^n)}{|(u_{i+1}^n - u_i^n)|} \leq \epsilon \tag{7} $$

This definition of wave speed ensures that shocks are captured with precision for any finite jump in \( u \), with \( \lambda_{i+\frac{1}{2}} \) assuming the Rankine-Hugoniot shock speed across a mesh interval. In this form the first order scheme appears as a central scheme and is comprised of a central difference in flux plus a central difference of a diffusion term. The scheme can be seen in its original upwind form by noting that for a positive wave speed the flux uses data to the left and reduces to \( f_{i+\frac{1}{2}} = F(u_i^n) \), otherwise \( f_{i+\frac{1}{2}} = F(u_{i+1}^n) \) and the flux uses data to the right. While the definition of Eq’s. (5-7) does not require any explicit sign dependence in the scheme, the upwind directions are clearly detected. This explicit scheme is the most fundamental scheme for scalar conservation laws in one dimension and is stable and monotonicity preserving subject to a maximum CFL condition of unity. This scheme also requires an entropy fix to disperse expansion shocks [8]. In order to apply such a scheme to a system of hyperbolic conservation laws the system is first decomposed into characteristic form via the transformation

$$ \Delta u = R \Delta v \tag{8} $$

where \( R \) is the matrix of right eigen-vectors of the system Jacobian matrix \( A = \partial F / \partial u \), the matrix of eigenvalues \( \Lambda \) is defined via

$$ AR = RA \tag{9} $$

and \( \Delta u, \Delta v \) represent the respective conservative and characteristic variable increments. The upwind scheme is in effect applied to each characteristic wave component and the discrete system is recomposed into conservation form. The first order scheme for a system is written as

$$ u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n) \tag{10} $$
where the approximate flux is defined by
\[ f^n_{i+\frac{1}{2}} = \frac{1}{2}(F(u^n_{i+1}) + F(u^n_i) - R|\Lambda_{i+\frac{1}{2}}|R^{-1}(u^n_{i+1} - u^n_i)) \] (11)
and the definition of the discrete eigenvalues \( \lambda_{i+\frac{1}{2}} \) now require an appropriate generalization of Eq.7 such that conservation together with the exact shock speed are maintained. This is the basis of the Roe scheme [2]. The CFL condition now applies with respect to the maximum eigenvalue of the system.

4 Approximate Riemann Solvers Without Upwinding in One Dimension

The appearance of the matrix of eigenvectors \( R \) in the system flux approximation is a consequence of upwinding on each characteristic component. This leads to an optimal diffusive operator in terms of numerical diffusion provided the shock jump criteria can be satisfied for the system in hand. If the matrix of eigenvalues is proportional to the unit matrix (equal eigenvalues) then the dependency of the discrete flux on the matrix of eigenvectors is removed leaving a much simpler diffusion coefficient.

The scheme can be simplified in this way without appearing to violate the crucial monotonicity preserving property of the scheme, subject to the CFL condition, by replacing the diagonal matrix of absolute eigen-values \( |\Lambda| \) evaluated at \( i + \frac{1}{2} \) with the matrix \( |\Lambda_{LF}| = |\Lambda_{LF;i+\frac{1}{2}}|I \) where
\[ |\lambda_{LF;i+\frac{1}{2}}| = \max_{xL,xR} \left( \max_j |\lambda^j_{i+\frac{1}{2}}| \right) \] (12)
and the maximum eigen-value in modulus of the system over the local interval \([x_i, x_{i+1}]\) is used. The discrete flux then takes the simpler form
\[ f^n_{i+\frac{1}{2}} = \frac{1}{2}(F(u^n_{i+1}) + F(u^n_i) - |\Lambda_{LF}|(u^n_{i+1} - u^n_i)) \] (13)
where it is understood that \( |\Lambda_{LF}| \) is defined on the interface \( i + \frac{1}{2} \). The flux of Eq. 13 is called the Local Lax-Friedrich flux LLF. The non-upwind LLF scheme is then defined be Eq’s. 13 and 10. Note that the scheme is still locally conservative. Extension to higher order accuracy is discussed in the next section. Since all eigenvalues in the diffusion operator are replaced by their maximum modulus, it follows that the price to be paid for this simplification is extra numerical diffusion. There is little discussion in the literature on the effect of the additional diffusion on the (low or higher order) results in terms of dissipation. There is a report that this scheme can be accompanied by further oscillations unless the global maximum eigenvalue over the domain is employed [6]. The next section presents an alternative non-upwind scheme to the LLF scheme, which relies on detecting the dominant wave of the system.
5 Dominant Wave of a System of Hyperbolic Conservation Laws

The notion of a dominant wave arises when attempting to describe a system of hyperbolic conservation laws with just a single best characteristic [7]. In order to illustrate the construction it is necessary to introduce the mobile operator that measures total rate of change in time along the characteristic as

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{Dx}{Dt} \frac{\partial u}{\partial x}
\]  

(14)

The system

\[
\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0
\]

(15)

can then be expressed in terms of the mobile operator as

\[
\frac{Du}{Dt} = \frac{Dx}{Dt} \frac{\partial u}{\partial x} - \frac{\partial F(u)}{\partial x}
\]

(16)

For a scalar equation the characteristic is defined by equating the right hand side of Eq. 16 to zero. In order to define the best single characteristic of a system, the \(L_2\) norm

\[
\| \frac{Du}{Dt} \|_2 = \| \frac{Dx}{Dt} \frac{\partial u}{\partial x} - \frac{\partial F(u)}{\partial x} \|_2
\]

(17)

of the total rate of change is instead minimised over \(Dx/Dt\). This leads to

\[
\frac{Dx}{Dt} = \frac{\partial u}{\partial x} \cdot \frac{\partial F(u)}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x}
\]

(18)

which in turn leads back to the exact characteristic for a scalar wave equation. This wave speed has been shown to be proportional to the dominant wave eigenvalue of the system [7]. This can be seen by invoking Eq’s. 8 and 9 in 18 to yield

\[
\frac{Dx}{Dt} = \left( \frac{\partial v^T}{\partial x} R^T R \frac{\partial v}{\partial x} \right) / \left( \frac{\partial v^T}{\partial x} R^T R \frac{\partial v}{\partial x} \right)
\]

(19)

which is therefore proportional to the eigenvalue corresponding to the strongest wave strength in characteristic gradient, and proves to be an optimal choice for a system when selecting a single wave speed.

6 Dominant Wave Scheme

The discrete dominant wave speed denoted here by \(\lambda_{DW}\) is naturally defined by

\[
\lambda_{DW} = \frac{((u_R - u_L) \cdot (F_R - F_L)) / ((u_R - u_L) \cdot (u_R - u_L))}
\]

(20)
which is a function of differences between left and right states \( L, R \) respectively. This definition is consistent with Eq. 7, for a scalar equation this definition reduces to the discrete wave speed of Eq. 7 for any finite jump in \( u \). An immediate advantage of this definition is the independence from a characteristic decomposition. In addition to detecting the system dominant wave, in the limit this definition equates to the exact shock speed for any wave type, which is a further property that is not possessed by other definitions of artificial viscosity coefficients, including the LLF scheme unless the Roe definition of eigenvalue is employed and the shock corresponds with the maximum eigenvalue. A discrete dominant wave flux is now defined by

\[
\mathbf{f}_{i+\frac{1}{2}}^n = \frac{1}{2} (F(u^n_{i+1}) + F(u^n_i) - |\Lambda_{DW}|(u^n_{i+1} - u^n_i))
\]

where

\[
|\Lambda_{DW}| = |\lambda_{DW}| J
\]

The dominant wave scheme is then defined be Eq’s. 21 and 10. This scheme is locally conservative and can be expected to retain stability whenever a particular wave dominates the system. The wave speed will adapt to the local dominant wave of the flow in different regions of the flow field as the solution evolves in time.

### 6.1 Dominant Wave Bounds

The definition of wave speed in Eq. 20 is singular unless there exists a non-zero finite jump or difference between left and right states. Numerical differentiation is one possibility in such cases and would retain complete independence from a characteristic decomposition since there would be no dependence on any knowledge of the system eigenvalues. Currently for certain cases bounds on \( \lambda_{DW} \) are imposed such that either

\[
\min_j |\lambda_j| \leq |\lambda_{DW}| \leq \max_j |\lambda_j|
\]

or specializing for the Euler equations with

\[
|u - a| \leq |\lambda_{DW}| \leq |u + a|
\]

The inequality of Eq. 23 is a property of Eq. 20 for a symmetric system Jacobian matrix, which suggests employing a symmetrized flux for calculating the wave speed (although this would still require numerical differentiation in singular cases). The above bounds ensure that the discrete dominant wave speed remains within the physical eigenvalue limits while also aiding stability of the method. Other possibilities include employing a smoother definition of wave speed constructed by e.g. a convex average between the absolute dominant wave and LLF with weighting as a function of e.g. Mach number.
7 Higher Order Schemes Without Upwinding in One Dimension

In the scalar case a higher order approximation is applied to the conservation variable. When an upwind scheme is applied to a system the higher order approximation is typically introduced wave by wave and applied to the characteristic variables, followed by recomposition to the conservative variables. In contrast, when using the discrete LLF flux Eq. 13, or the dominant wave flux defined by Eq. 21 respectively, the extension to higher order accuracy is simpler to achieve with the non-upwind formulations when applied to systems, since there is no dependency upon characteristic variables in these definitions of flux. Consequently a higher order approximation can be introduced for the left and right states respectively and be expressed directly in terms of the conservative variables. Alternatively the higher order expansions can also be applied to other sets of variables such as primitive or characteristic variables. In this work the conservative variables are used directly.

In one dimension the scheme is expressed as a two-step process. First the higher order states are defined using a MUSCL formalism [9]. Higher order left and right hand side states are obtained by expansions about the states L and R, viz

\[
\begin{align*}
\mathbf{u}_{L_{i+\frac{1}{2}}} & = \mathbf{u}_i + \frac{1}{2}\Phi(r_{i+1/2}^+)(\mathbf{u}_{i+1} - \mathbf{u}_i) \\
\mathbf{u}_{R_{i+\frac{1}{2}}} & = \mathbf{u}_{i+1} - \frac{1}{2}\Phi(r_{i+1/2}^-)(\mathbf{u}_{i+1} - \mathbf{u}_i)
\end{align*}
\]

(25)

where \(\Phi(r_{i+1/2}^+\) and \(\Phi(r_{i+1/2}^-\) are flux limiters, or in this case slope limiters [9]. The slope limiters are functions of adjacent discrete gradients where \(r_{i+1/2}^+ = (\Delta u_i - \Delta u_{i+1})/\Delta u_i\) and \(r_{i+1/2}^- = (\Delta u_{i+1} - \Delta u_i)/\Delta u_i\) and \(\Delta u_{i+1} = \mathbf{u}_{i+1} - \mathbf{u}_i\). The slope limiters constrain the expansions to ensure that the higher order data remains monotonic. Details can be found in [9] and [10]. The above fluxes of Eq.’s 13, 21 can now be applied to the higher order data so that the local Riemann problem is resolved with a higher order flux of the form

\[
\mathbf{f}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}(\mathbf{u}_{R_i}^{n+\frac{1}{2}}) + \mathbf{F}(\mathbf{u}_{L_i}^{n+\frac{1}{2}})) - |\Lambda_D|(\mathbf{u}_{R_i}^{n+\frac{1}{2}} - \mathbf{u}_{L_i}^{n+\frac{1}{2}}))
\]

(26)

where \(\Lambda_D = \Lambda_{LF}\) corresponds to the LLF scheme and \(\Lambda_D = \Lambda_{DW}\) corresponds to dominant wave scheme respectively. Either flux can now be used to integrate the system via Eq.10. The higher order spatial scheme as defined by Eq.’s 25, 26 and 10, uses first order forward-Euler time stepping. The CFL condition of the scheme is dependent on the choice of limiter, typically an upper limit of 1/2 is selected. Extension to higher order time accuracy can be achieved with the second or third order Runge-Kutta schemes of [14] which preserve monotonicity. The CFL limit reduces further in higher dimensions. Note that the first order flux is recovered if the limiters are set to zero.

8 Approximate Riemann Solvers Without Upwinding in Two Dimensions

The higher dimensional extension of the above scheme is based on a direct generalization of the one dimensional discrete flux. The flow equations Eq. 1 are integrated
in space and time over a discrete control-volume \( \Omega \) with surface \( \delta \Omega \) by direct use of the Gauss divergence theorem applied to yield a surface integral of divergence

\[
\int_\Omega (\mathbf{u}(t + \Delta t) - \mathbf{u}(t)) dV = \int_{\Delta t} \int_{\delta \Omega} (\mathbf{F} dy - \mathbf{G} dx) dt
\]  

(27)

Discrete cell vertex approximations are developed for general unstructured grids comprised of quadrilateral and-or triangular cells. A control-volume is constructed around each grid vertex, by joining centres of cell edges that pass through a given vertex to centres of the cells that share the common vertex. We shall denote the \( i^{th} \) vertex control-volume by \( \Omega_i \) and surface \( \delta \Omega_i \). Referring to the typical control-volume shown in Fig. 1. The surface outward normal increment of the \( p^{th} \) face of \( \delta \Omega_i \) is denoted by \( \Delta n_p = (\Delta y_p, -\Delta x_p) \). The approximation of the surface integral of a flux function \((\mathbf{F}(\mathbf{u}), \mathbf{G}(\mathbf{u}))\) over the control-volume is defined by the sum of flux increments over each face

\[
\sum_{p}^{N_f} \mathcal{F}(\mathbf{u}_p) |\Delta n_p| = \sum_{p}^{N_f} (\mathbf{F}(\mathbf{u}_p) \Delta y_p - \mathbf{G}(\mathbf{u}_p) \Delta x_p)
\]  

(28)

where summation is over the surrounding \( N_f \) control-volume faces and the face-normal flux is defined by

\[
\mathcal{F}(\mathbf{u}) = \frac{\mathbf{F}(\mathbf{u}) \Delta y - \mathbf{G}(\mathbf{u}) \Delta x}{|\Delta n|}
\]  

(29)

As in one dimension the scheme is expressed as a two-step process: First discrete fluxes are defined on the control-volume faces, which are assembled in an edge-wise fashion. In the case of the first order scheme the first step simply involves selecting the left and right hand side states, which correspond to the left and right hand side edge vertex values of the conservation variables. The generalization of Eq. 26 is employed with the directionally resolved outward normal flux of Eq. 29 replacing the one-dimensional flux evaluated at the respective left and right hand states, so that edge based Riemann problems are resolved with the discrete generalised flux

\[
f(\mathbf{u}_R, \mathbf{u}_L) = \frac{1}{2} (\mathcal{F}(\mathbf{u}_R^n) + \mathcal{F}(\mathbf{u}_L^n) - |\Lambda_D|(\mathbf{u}_R^n - \mathbf{u}_L^n))
\]  

(30)

and as before \( \Lambda_D = \Lambda_{LF} \) and \( \Lambda_D = \Lambda_{DW} \) correspond to the LLF and dominant wave schemes respectively. For the Euler equations the LLF eigenvalue is defined by

\[
\lambda_{LF} = \max_{L,R} (|\mathbf{q} \cdot \hat{\mathbf{n}}| + a)
\]  

(31)

For the dominant wave scheme, the dominant wave speed of Eq 20 is now applied using the normally resolved flux of Eq. 29 and the speed is a function of left and right edge-vertex states,
\[ \lambda_{DW} = \frac{((u_R - u_L) \cdot (F(u)_R - F(u)_L))}{((u_R - u_L) \cdot (u_R - u_L))} \] (32)

Note that this definition applies to any hyperbolic system. In two dimensions bounds analogous to Eq’s. 23-24 are invoked.

The second step of the non-upwind scheme formulation involves formal time integration of the fully discrete divergence of flux for each component of the system, which is shown below with forward-Euler time integration. The discrete scheme for vertex \( i \) (control-volume area \( \Delta A_i \)) can then be written as

\[ u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{\Delta A_i} \sum_{p} f_{p}(u_R, u_L)|\Delta n_p| \] (33)

which completes the discrete approximation of Eq. 27.

9 Higher Order Schemes Without Upwinding in Two Dimensions

Higher order states are defined using a MUSCL formalism [9], which in essence follows the above 1-D principle applied relative to each edge along which flux is to be defined. The higher order left and right hand side states are obtained by expansions about the vertex locations at \( i \) and \( j \), Fig.2. As in 1-D the expansions are constrained with slope limiters to ensure that the higher order data remains monotonic. Referring to Fig.2 the left and right states L and R of edge \( e \) are expressed as

\[ u_L = u_i + \frac{1}{2} \Phi(r_{i,j}^+) \Delta u_{i,j} \] (34)

\[ u_R = u_j - \frac{1}{2} \Phi(r_{i,j}^-) \Delta u_{i,j} \]

where

\[ r_{i,j}^+ = \left( \Delta u_{u,i} / \Delta u_{i,j} \right) \] (35)

\[ r_{i,j}^- = \left( \Delta u_{j,\bar{d}} / \Delta u_{i,j} \right) \]

where \( \Delta u_{i,j} = u_j - u_i \) is the edge difference, however the differences \( \Delta u_{u,i} \) and \( \Delta u_{j,\bar{d}} \) are only well defined on a structured grid, where the location of \( u_{u,i} \) and \( u_{j,\bar{d}} \) would correspond to the next upstream and downstream nodes of the grid. Extension to unstructured grids requires special construction of the differences \( \Delta u_{u,i} \) and \( \Delta u_{j,\bar{d}} \). Two types of definition are considered here for unstructured grids. The first is based directly on [11] with a generalised central gradient defined at the vertex by \( \nabla u_i \). In this case the gradient of the function is resolved along the direction of the edge and the upstream difference determined from the vertex gradient and edge difference assuming a locally resolved central difference relationship

\[ \Delta u_{u,i} = 2\nabla u_i \cdot dr_{i,j} - \Delta u_{i,j} \] (36)
with a similar definition for the downstream difference involving the edge difference and gradient at vertex j. While this scheme proves to be quite effective, the scheme lacks formal monotonicity and oscillations can occur in the vicinity of strong shocks, the scheme requires the addition of a maximum-minimum constraints e.g. as proposed by [12]. Further possibilities are presented in [13].

The second definition involves a direct extrapolation of the respective upstream and downstream data such that monotonicity holds. The edge vector is extrapolated upstream (and down stream) see arrows in Fig.2 and a convex average interpolant of data is defined for the two (bold) triangles containing the arrows: e.g. for vertex i

$$\Delta u_{i,i} = (1 - \xi)\Delta u_{u1,i} + \xi\Delta u_{u2,i}$$

where $\xi$ is a function of the angle of entry of the vector into the triangle, similarly for vertex j. This scheme is essentially Local Edge Diminishing LED in motivation [3], [13], but applied to the data. This completes the definition of the higher order states. The second step of the scheme now uses cf. Eq.’s 34 within 33 with either LLF Eq. 12, 31 or the dominant-wave Eq. 22 and 32. As in 1-D the Runge-Kutta schemes of [14] can be used to complete the formal accuracy of the scheme without disturbing monotonicity subject to a CFL constraint, here 1/4 is used. As is typical of flux limited schemes, convergence to steady state is sensitive and limiting is based on the van-Leer (smooth) limiter

$$\Phi(r) = \frac{2r}{1 + r}$$

Note as before, that the first order flux is recovered if the limiters are set to zero.

10 RESULTS

10.1 One Dimension

The schemes are first compared for the shock tube problem [15] (exact solution solid line, computed square symbol, time = 0.165) with the prescribed initial data

$$(\rho, u, p) = \begin{cases} 1.0, & 0.0, & 1.0 \quad x \leq 0.5 \\ 0.125, & 0.0, & 0.1 \quad x > 0.5 \end{cases}$$

The higher order scheme results are shown in Fig’s. 3 - 4 using 50 nodes. The comparison for density clearly shows the additional diffusion that is inherent in the LLF scheme Fig. 3, where the contact discontinuity is completely smeared by the diffusion of the scheme compared to the dominant wave scheme Fig. 4, which improves the resolution of the contact discontinuity. Both schemes show a significant improvement in resolution using 100 nodes Fig’s. 5 - 6, consistent with high resolution scheme performance. In this case a 3-shock forms, corresponding to the maximum eigenvalue, so that the LLF scheme is well suited to the problem. However, the dominant wave scheme still provides a better overall result due to the improvement in resolution in the region of the contact.
10.2 Two Dimensions

Results are presented for two types of test case in two dimensions. The first test is the well known transonic flow over a circular arc, with 10 percent radius, for details see e.g. [17]. The initial free stream mach number is specified as 0.675. Subsonic inflow and outflow boundary conditions apply. Data is prescribed according to the number of inward pointing characteristics. Density is updated on the inflow boundary (only one outward pointing characteristic) and the pressure is prescribed on the outflow boundary (only one inward pointing characteristic). A weak shock forms on the arc with the shock-foot at 72 percent chord [17]. The triangular grid used for the computational comparisons is shown in Fig. 7. The results are presented in the form of Mach number contours. The higher order LLF scheme results are shown in Fig. 8. The diffusive nature of the LLF scheme is clearly apparent for this problem, where the weak 1-shock proves to be too weak for the scheme to detect its formation. The additional diffusion based on the largest eigenvalue is clearly inappropriate for such problems, which are actually governed by the smallest eigenvalue of the system. The LLF contours only indicate a slight asymmetry in the region of the shock, despite the use of a higher order scheme. In contrast the Dominant-Wave scheme results of Fig. 9 clearly demonstrate that the new scheme is applicable to transonic flow and can detect the dominant (minimum) eigenvalue in this case, resulting in capture of the shock over the mesh interval that straddles 72 percent chord.

The second case involves supersonic flow over a wedge with 20 degree angle. The initial free stream mach number is specified as 3.0. Supersonic inflow and outflow boundary conditions apply with the vector \( \mathbf{u} \) prescribed on inflow boundaries and the solution vector updated on outflow boundaries. The exact solution is comprised of a strong shock which forms at the corner of the wedge at an angle of 37.5 degrees [16]. The triangular grid used for the computational comparisons is shown in Fig. 10. The results are presented in the form of density contours.

The higher order LLF scheme results are shown in Fig. 11. The LLF scheme is able to resolve the shock in this case, due to the stronger self sharpening nature of the shock. However, the diffusive nature of the LLF scheme is still apparent for this problem with some visible spreading of contours. The additional diffusion inherent in the scheme arises due to using the modulus of the largest eigenvalue to govern stability of all wave components throughout the field. The Dominant-Wave scheme results are shown in Fig. 12 and clearly demonstrate an improvement in shock resolution for supersonic flow. This is due to the ability of the new schemes to adjust towards the local dominant wave eigenvalue of the flow, resulting in capture of the shock with less numerical diffusion. The centre line of the band of contours is in line with the exact solution.
Figure 1: Control-Volume

Figure 2: Higher Order Edge-based Flux Support (in Bold)
Figure 3: Shock-tube 50 nodes: Higher Order Lax-Friedrichs

Figure 4: Shock-tube 50 nodes: Higher Order Dominant-Wave
Figure 5: Shock-tube 100 nodes: Higher Order Lax-Friedrichs

Figure 6: Shock-tube 100 nodes: Higher Order Dominant-Wave
Figure 7: Channel: Grid

Figure 8: Mach Contours: Higher Order Lax-Friedrichs

Figure 9: Mach Contours: Higher Order Dominant-Wave
Figure 10: Wedge: Triangular Grid

Figure 11: Density Contours: Higher Order Lax-Friedrichs

Figure 12: Density Contours: Higher Order Dominant-Wave
11 CONCLUSIONS

This paper presents a new dominant wave capturing formulation for hyperbolic conservation laws. The method is constructed so that local conservation is maintained and is developed within a general finite volume framework. Low order and higher order versions of the method are presented on structured and unstructured grids.

Comparisons between the new method and the LLF schemes reveal clear advantages of the new dominant wave formulation.

The schemes are applied to the Euler equations of compressible flow. Results are shown for some classical flow problems. The LLF schemes can either fail to detect discontinuities that are known to be present in the physical solution, or introduce larger amounts of numerical diffusion. In contrast, the new dominant wave scheme is able to capture the discontinuities while using exactly the same grids and equivalent levels of accuracy in terms of polynomial approximation. The results presented demonstrate the benefits of the dominant wave formulation, and show that improved resolution can be obtained compared to the LLF formulation while retaining stability, producing essentially monotonic solutions and continuing to circumvent the need for a characteristic decomposition. The improvement is attributed to the ability of the new scheme to detect the crucial dominant wave eigenvalue of the system and thereby reduce the global numerical diffusion that is added by the LLF schemes dependence on the maximum system eigenvalue. This is demonstrated for higher order approximations on structured and unstructured grids.

Finally it is noted that the dominant wave formulation offers the benefits of being directly applicable to other systems of hyperbolic conservation laws without requiring a characteristic decomposition.

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REFERENCES


