THE TOPOLOGICAL-SHAPE SENSITIVITY METHOD AND ITS APPLICATIONS IN TOPOLOGY DESIGN AND INVERSE PROBLEMS

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Abstract. The topological derivative gives the sensitivity of a cost function when the domain under consideration is perturbed by the introduction of a hole. In particular, the topological derivative was already applied for solving topology design of several engineering problems. Alternatively, this same idea can also be used to calculate the sensitivity of the problem when, instead of a hole, a small incrustation is introduced at a point in the domain. Therefore, the topological derivative concept is wider. In fact, it also can be applied to inverse problems and to simulate physical phenomena with changes on their configuration. Thus, in the present paper, the topological derivative computed through the novel Topological-Shape Sensitivity Method is applied in a to 2D and 3D heat conduction, torsion with creep and inverse conductivity problem.
1 INTRODUCTION

The topological derivative leads to a scalar function that supplies, for each point of the domain under consideration, the sensitivity of a given cost function when a small hole is created [5, 23, 3, 10, 15]. More recently, in [19, 9, 8] a new method to compute the topological derivative via shape sensitivity analysis was proposed. This method, called Topological-Shape Sensitivity Method has two main features. First, leads to a very simple and general procedure to compute the topological derivative. Second, allow us to consider several kind of cost functions and any type of boundary conditions on the hole.

It has been already accepted that topological derivative furnishes a powerful tool for topology optimization (see [4], where 425 references concerning topology optimization of continuum structures are included). Nevertheless, this concept is wider. In fact, the same theory developed for topological derivative can be used to calculate the sensitivity of a given cost function when, instead of a hole, a small incrustation is introduced at a point in the domain. This concept called configurational derivative [17] can be naturally applied in the inverse problems context (see, for instance, [14]). Among others, on identification of defects in mechanical components and properties characterization in heterogeneous media.

In order to show the applicability of the Topological-Shape Sensitivity Method as a systematic methodology for computing the topological derivative, we apply this novel approach in several engineering problems. More specifically, we compute the topological derivative in 2D and 3D steady-state heat conduction (section 3.1), torsion of steady-state creep shafts (section 3.2) and inverse Poisson’s conductivity problem (section 3.3).

2 TOPOLOGICAL-SHAPE SENSITIVITY METHOD

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) with a smooth boundary $\partial \Omega$. If the domain $\Omega$ is perturbed by introducing a small hole at an arbitrary point $\hat{x} \in \Omega$, we have a new domain $\Omega_\epsilon = \Omega - \overline{B}_\epsilon$, whose boundary is denoted by $\partial \Omega_\epsilon = \partial \Omega \cup \partial B_\epsilon$, where $\overline{B}_\epsilon = B_\epsilon \cup \partial B_\epsilon$ is a ball of radius $\epsilon$ centered at the point $\hat{x} \in \Omega$. Thus, we have the original domain without hole $\Omega$ and the new one $\Omega_\epsilon$ with a small hole $B_\epsilon$. Considering a cost function $\psi$, the topological derivative is defined as [10]

$$D_T(\hat{x}) := \lim_{\epsilon \to 0} \frac{\psi(\Omega_\epsilon) - \psi(\Omega)}{f(\epsilon)},$$

where $f(\epsilon)$ is a negative function that decreases monotonically so that $f(\epsilon) \to 0$ with $\epsilon \to 0^+$.

Several methods have been proposed to compute the topological derivative [23, 3, 10]. In [19, 8, 21] the authors proposed an alternative approach to calculate this derivative, called Topological-Shape Sensitivity Method, which allows us to use the whole mathematical framework (and results) developed for the shape sensitivity analysis (see [2, 11, 12, 16, 22, 24, 25, 26] and references therein). This method is based on the link between the shape and topology derivatives that is given by the following theorem:
**Theorem 1** Let \( f(\epsilon) \) be a function chosen in order to \( 0 < |D_T(\bar{X})| < \infty \), then the topological derivative given by eq. (1) can be written as

\[
D_T(\bar{X}) = \lim_{\epsilon \to 0} \frac{1}{f'(\epsilon)} \frac{d}{d\tau} \psi(\Omega_\tau) \bigg|_{\tau=0},
\]

where \( \tau \in \mathbb{R}^+ \) is used to parameterize the domain. That is, for \( \tau \) small enough, we have

\[
\Omega_\tau := \{ x_\tau \in \mathbb{R}^N \mid \exists \ x \in \Omega_\epsilon, \ x_\tau = x + \tau v, \ x_\tau|_{\tau=0} = x \ \text{and} \ \Omega_\tau|_{\tau=0} = \Omega_\epsilon \},
\]

being \( v \) the shape change velocity defined by

\[
v = -n \ \text{on} \ \partial B_\epsilon \ \text{and} \ v = 0 \ \text{on} \ \partial \Omega. \quad (3)
\]

In addition,

\[
\frac{d}{d\tau} \psi(\Omega_\tau) \bigg|_{\tau=0} = \lim_{\tau \to 0} \frac{\psi(\Omega_\tau) - \psi(\Omega_\epsilon)}{\tau}.
\]

is the shape sensitivity of the cost function in relation to the domain perturbation characterized by \( v \).

**Proof.** The reader interested in the proof of this result may refer to [19, 8, 21] $
$

This theorem points out that the topological derivative may be obtained through the shape sensitivity analysis of the cost function (Topological-Shape Sensitivity Method). Therefore, results from shape sensitivity analysis can be used to calculate the topological derivative in a simple and constructive way considering eq. (2).

It is important to mention that in general the cost function \( \psi(\Omega_\epsilon) \) also depends on a function \( u_\epsilon \in U_\epsilon \) which is the solution of a variational problem stated as: find \( u_\epsilon \in U_\epsilon \), such that

\[
a_\epsilon(u_\epsilon, \eta) = l_\epsilon(\eta) \quad \forall \eta \in V_\epsilon,
\]

where \( U_\epsilon \) and \( V_\epsilon \) respectively are the sets of admissible functions and admissible variations defined on the domain \( \Omega_\epsilon \), which will be characterized later according to the problem under analysis, and the operator \( a_\epsilon(\cdot, \cdot) : U_\epsilon \times V_\epsilon \to \mathbb{R} \) is a bilinear form and \( l_\epsilon(\cdot) : V_\epsilon \to \mathbb{R} \) is a linear functional.

In the same way, the state equation written in the reference configuration (eq. 5), must also be satisfied in the perturbed configuration \( \Omega_\tau \), \( \forall \tau \geq 0 \). Therefore, function \( u_\tau \in U_\tau \) may satisfies the following variational problem: find \( u_\tau \in U_\tau \), such that

\[
a_\tau(u_\tau, \eta) = l_\tau(\eta) \quad \forall \eta \in V_\tau,
\]

where \( a_\tau(\cdot, \cdot) : U_\tau \times V_\tau \to \mathbb{R} \), \( l_\tau(\cdot) : V_\tau \to \mathbb{R} \) and \( U_\tau, V_\tau \) are the sets of admissible functions and admissible variations, respectively, defined on the domain \( \Omega_\tau \). Finally, the cost function \( \psi(\Omega_\tau) := J_\tau(u_\tau) \).
Formally, the derivative of $J_r(u_r)$ in relation to the parameter $\tau$ at $\tau = 0$ used to compute the topological derivative (eq. 2) reads

$$\left\{
\begin{array}{l}
\text{Calculate} : \quad \frac{d}{d\tau} J_r(u_r) |_{\tau=0} \\
\text{Subject to} : \quad a_r(u_r, \eta) = l_r(\eta) \quad \forall \eta \in \mathcal{V}_r \text{ and } \forall \tau \geq 0 .
\end{array}\right.$$ (7)

The shape derivative of the cost function considering the state equation as the constraint (eq. 7) can be computed using the Lagrangian method that consists in relaxing the constraint of the problem by Lagrangian multipliers. Therefore, the Lagrangian already defined in $\Omega_\epsilon$ is written as

$$L_\tau(v, \mu) = J_r(v) + a_r(v, \mu) - l_r(\mu) \quad \forall \mu \in \mathcal{V}_r \text{ and } v \in \mathcal{U}_r .$$ (8)

Let us consider $v = u_\tau \in \mathcal{U}_r$ solution of the state equation (eq. 6) and $\mu = \lambda_\tau \in \mathcal{V}_r$ solution of the adjoint equation given by: find $\lambda_\tau \in \mathcal{V}_r$, such that

$$a_r(\lambda_\tau, \eta) = -\left(\frac{\partial}{\partial \lambda_\tau} J_r(u_r), \eta\right) \quad \forall \eta \in \mathcal{V}_r .$$ (9)

Then, we have the following well-known result

$$\frac{d}{d\tau} J_r(u_r) = \frac{\partial}{\partial \tau} L_r(u_r, \lambda_\tau) = \frac{\partial}{\partial \tau} J_r(u_r) + \frac{\partial}{\partial \tau} a_r(u_r, \lambda_\tau) - \frac{\partial}{\partial \tau} l_r(\lambda_\tau) .$$ (10)

3 APPLICATIONS IN ENGINEERING PROBLEMS

In this paper we apply the Topological-Shape Sensitivity Method for computing the topological derivative in 2D and 3D steady-state heat conduction (section 3.1), torsion of steady-state creep shafts (section 3.2) and inverse Poisson’s conductivity problem (section 3.3).

3.1 Steady-state heat conduction

Let a rigid body be represented by $\Omega_\epsilon \subset \mathbb{R}^2$ submitted to a constant excitation $b$ in the domain $\Omega_\epsilon$ and Dirichlet boundary condition on $\partial \Omega_\epsilon$. Considering that on $\partial B_\epsilon$ we have Dirichlet, Neumann or Robin boundary conditions, then the solution $u_\epsilon$ must satisfies the following variational equation: find $u_\epsilon \in \mathcal{U}_\epsilon$, such that

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla \eta + \gamma \int_{\partial B_\epsilon} u_\epsilon \eta = \int_{\Omega_\epsilon} b \eta + (\beta + \gamma) \int_{\partial B_\epsilon} h \eta \quad \forall \eta \in \mathcal{V}_\epsilon ,$$ (11)

where $\mathcal{U}_\epsilon$ and $\mathcal{V}_\epsilon$ are given, respectively, by

$$\mathcal{U}_\epsilon = \{ u_\epsilon \in H^1(\Omega_\epsilon) \mid u_\epsilon|_{\partial \Omega} = \bar{u} , \alpha (u_\epsilon - h) = 0 \text{ on } \partial B_\epsilon \} ,$$

$$\mathcal{V}_\epsilon = \{ \eta \in H^1(\Omega_\epsilon) \mid \eta|_{\partial \Omega} = 0 , \alpha \eta = 0 \text{ on } \partial B_\epsilon \} ,$$

and $\alpha, \beta, \gamma \in \{0, 1\}$, with $\alpha + \beta + \gamma = 1$. This notation should be interpreted as follows: when $\alpha = 1$, $u_\epsilon = h$ and $\eta = 0$ on $\partial B_\epsilon$, and when $\alpha = 0$, $u_\epsilon$ and $\eta$ are free on $\partial B_\epsilon$. 
3.1.1 Shape sensitivity analysis

Let us consider a cost function only depending implicitly on \( \Omega_\epsilon \) through the solution \( u_\epsilon \). Thus, the cost function \( \psi \) can be written as \( \psi (\Omega_\epsilon) := \mathcal{J} (u_\epsilon) \). Therefore, from the Reynolds’ transport theorem and considering the Lagrangian method, the derivative of the cost function \( \psi (\Omega_\epsilon) := \mathcal{J} (u_\epsilon) \) at \( \tau = 0 \), becomes

\[
\frac{d}{d\tau}\mathcal{J}(u_\tau)\bigg|_{\tau=0} = \int_{\Omega_\epsilon} \Sigma_\epsilon \cdot \nabla \mathbf{v} + \int_{\partial B_\epsilon} \left( \gamma (u_\epsilon - h) - \beta h \right) \lambda_\epsilon \text{div}_{\partial \Omega} \mathbf{v},
\]

where \( \text{div}_{\partial \Omega} \mathbf{v} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla \mathbf{v} \) is the superficial divergent of the velocity \( \mathbf{v} \) and \( \Sigma_\epsilon \) can be interpreted as a generalization the Eshelby’s energy-momentum tensor (see, for instance [6, 25]), is given by

\[ \Sigma_\epsilon = (\nabla u_\epsilon \cdot \nabla \lambda_\epsilon - b \lambda_\epsilon) \mathbf{I} - (\nabla u_\epsilon \otimes \nabla \lambda_\epsilon + \nabla \lambda_\epsilon \otimes \nabla u_\epsilon). \]

3.1.2 Topological derivative calculation

Taking into account that the Eshelby tensor given by eq. (13) has null-divergence and considering the definition of the velocity field (eq. 3) and Theorem 1 (eq. 2), the topological derivative results in an integral defined on the boundary of the hole \( \partial B_\epsilon \), that is

\[
D_T (\hat{x}) = \lim_{\epsilon \to 0} \frac{1}{f'(\epsilon)} \int_{\partial B_\epsilon} \left[ \frac{\partial u_\epsilon}{\partial t} \frac{\partial \lambda_\epsilon}{\partial t} - \frac{\partial u_\epsilon}{\partial n} \frac{\partial \lambda_\epsilon}{\partial n} - b \lambda_\epsilon - \frac{1}{\epsilon} \left( \gamma (u_\epsilon - h) - \beta h \right) \lambda_\epsilon \right].
\]

In order to obtain the final expression of the topological derivative, we need to explicitly know the behavior of the solutions \( u_\epsilon \) and \( \lambda_\epsilon \) for \( \epsilon \to 0 \), as well as their normal and tangential derivatives. Thus, an asymptotic analysis of \( u_\epsilon \) and \( \lambda_\epsilon \) shall be performed [17, 13]. Therefore, from this analysis, it is possible to compute the limit \( \epsilon \to 0 \) in eq. (14) obtaining the results presented in Table 1, where \( u \) and \( \lambda \) are the solutions of the state and adjoint equations, respectively, both defined in the original domain \( \Omega \) (without hole).

<table>
<thead>
<tr>
<th>boundary conditions</th>
<th>( f(\epsilon) )</th>
<th>( D_T j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta = 1, \alpha = \gamma = 0 ) and ( h \neq 0 )</td>
<td>(-2\pi \epsilon)</td>
<td>( h \lambda )</td>
</tr>
<tr>
<td>( \beta = 1, \alpha = \gamma = 0 ) and ( h = 0 )</td>
<td>(-\pi \epsilon^2)</td>
<td>( 2 \nabla u \cdot \nabla \lambda - b \lambda )</td>
</tr>
<tr>
<td>( \gamma = 1, \alpha = \beta = 0 )</td>
<td>(-2\pi \epsilon)</td>
<td>(- (u - h) \lambda )</td>
</tr>
<tr>
<td>( \alpha = 1, \beta = \gamma = 0 ) and ( h = h^* )</td>
<td>(-2\pi \epsilon^2)</td>
<td>(-2 \nabla u \cdot \nabla \lambda )</td>
</tr>
<tr>
<td>( \alpha = 1, \beta = \gamma = 0 ) and ( h \neq h^* )</td>
<td>( \frac{2\pi}{\log(\epsilon)})</td>
<td>(- (u - h) \lambda )</td>
</tr>
</tbody>
</table>
Remark 2 The exceptional case \( h = h^* \) appears in the Saint-Venant theory of torsion of elastic shafts, that is

\[
\int_{\partial B_c} \frac{\partial u_c}{\partial n} = b \pi \epsilon^2 .
\]  

In addition, we can use the same development to compute the topological derivative in three dimensional Poisson’s equation. In fact, considering Dirichlet or Neumann boundary conditions on \( \partial B_c \) (both homogeneous, that is, for \( h = 0 \)), the topological derivative, for \( b = 0 \), can be summarized in Table 2, remembering that \( u \) and \( \lambda \) are the solutions of the state and adjoint equations, respectively, both defined in the original domain \( \Omega \) (without hole).

**Table 2:** Topological derivatives for Poisson’s problem in 3D domains.

<table>
<thead>
<tr>
<th>boundary conditions</th>
<th>( f (\epsilon) )</th>
<th>( DT )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta = 1, \alpha = 0 ) and ( h = 0 )</td>
<td>(-\frac{4}{3} \pi \epsilon^3)</td>
<td>(\frac{4}{3} \nabla u \cdot \nabla \lambda)</td>
</tr>
<tr>
<td>( \alpha = 1, \beta = 0 ) and ( h = 0 )</td>
<td>(-4 \pi \epsilon)</td>
<td>(-u \lambda)</td>
</tr>
</tbody>
</table>

### 3.1.3 Additional comments

In this section the topological derivative was calculated for Poisson’s problem in 2D \((N = 2)\) domains using Topological-Shape Sensitivity Method. For the sake of simplicity, we have adopted a cost function that depends only implicitly on the domain. However, we have considered a general set of boundary conditions on the holes: Dirichlet, Neumann (both homogeneous and non-homogeneous), Robin and the exceptional case associated to the Saint-Venant theory of torsion of elastic shaft. Furthermore, we have computed the topological derivative for Poisson’s problem in 3D \((N = 3)\) domains considering homogeneous Dirichlet and Neumann boundary conditions on the holes. Finally, we have shown a summary of the results in Table 1 for 2D and Table 2 for 3D.

### 3.2 Torsion of steady-state creep shafts

In this section, we study the topological derivative for a non-linear case: torsion of steady-state creep shafts. The non-linearity of this problem arises from assuming steady-state creep behavior of the power-law type \([1]\). Considering Norton’s material and following Prandtl’s approach the variational formulation of the problem may be stated as: find \( u_c \in \mathcal{U}_c \), such that

\[
\int_{\Omega_c} 3k \left( 3 \nabla u_c \cdot \nabla u_c \right)^{\frac{p-2}{2}} \nabla u_c \cdot \nabla \eta = 2b \left( \int_{\Omega_c} \eta + \int_{B_c} \eta \right) \quad \forall \ \eta \in \mathcal{V}_c ,
\]  

(16)
where \( k \) and \( p \) are material constants and \( \mathcal{U}_e, \mathcal{V}_e \) are defined as
\[
\mathcal{V}_e = \mathcal{U}_e = \{ u_e \in W^{1,p}(\Omega_e) \mid u_e|_{\partial\Omega} = 0, \quad u_e|_{\partial B_c} = \overline{\tau} \}.
\]
In addition, \( \overline{\tau} \) and \( \overline{\tau} \) are arbitrary constants defined on \( \partial B_e \) and \( b \) is a rigid rotation of the cross-section of the bar. It is important to observe that eq. (16) represents the weak formulation of the \( p \)-Poisson’s equation.

### 3.2.1 Shape sensitivity analysis

Considering the complementary dissipation energy as the cost function, \( \psi(\Omega_e) := J_e(u_e) \) can be defined by
\[
J_e(u_e) := \frac{1}{p} \int_{\Omega_e} k (3\nabla u_e \cdot \nabla u_e)^p - 2b \left( \int_{\Omega_e} u_e + \int_{B_c} \bar{u}' \right).
\]
(17)

If \( u_e \) is the solution of the state equation associated to the domain \( \Omega_e \), then the shape derivative of the cost function given by eq. (17) at \( \tau = 0 \) becomes
\[
\frac{d}{d\tau} J_e(u_e) \bigg|_{\tau=0} = \int_{\Omega_e} \Sigma_e \cdot \nabla v - 2b\overline{\tau}' \int_{B_c} \text{div} v,
\]
where \( \Sigma_e \) is the Eshelby energy momentum tensor given, in this particular case, by
\[
\Sigma_e = \left[ \frac{k}{p} (3\nabla u_e \cdot \nabla u_e) \right] - 2b \int_{B_c} (3\nabla u_e \cdot \nabla u_e)^{p-2} (\nabla u_e \otimes \nabla u_e).
\]
(19)

### 3.2.2 Topological derivative calculation

Taking into account that \( \text{div} \Sigma_e = 0 \) (eq. 19) and considering Theorem 1 (eq. 2) for the velocity field defined by eq. (3), the topological derivative results in an integral only defined on the boundary of the hole \( \partial B_e \), that is
\[
D_T(\hat{x}) = - \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\partial B_c} \frac{1}{p} \frac{p}{3^2 k} \left( \frac{\partial u_e}{\partial n} \right)^p \text{ },
\]
(20)
where we have also considered the boundary condition on \( \partial B_e \).

In order to compute the limit \( \epsilon \to 0 \) in eq. (20), we need to know the behavior of \( u_e \) in the neighborhood of the ball \( B_e \). However, this problem is non-linear and this limit is not trivial to be analytically computed. Therefore, we will perform a numerical study of the behavior of the integrand of eq. (20) when \( \epsilon \) takes small values.

Let us consider a straight bar of square cross-section \((0,2) \times (0,2)\) under uniform twist \( b = 1 \) with a hole \( \epsilon = \{0.01, 0.02, 0.04, 0.08, 0.16\} \), such as the one shown in Fig. 1, where \( a = 0.75 \) and \( c = 0.5 \). The material properties of the bar are \( k = 1 \) and \( p = \{2, 3, 4, 5, 6\} \).
We analyze this problem using the quadratic *simplex* finite element. The mesh is constructed maintaining the same number of elements on the boundary of the hole for whichever radius $\epsilon$. Thus, we use the following relation to estimate the size of the elements $h^e$
\[
h^e \approx \frac{2\pi r}{ne},
\]
where $ne$ is the required number of elements on the boundary of the hole (for $r = \epsilon$) and $r$ is the radius from the center of the hole (see Fig. 1). In Table 3 we show the mesh used to analyze the problem with hole.

Let us also consider a function $d_T(u_\epsilon)$ such that (see eq. 20)
\[
d_T(u_\epsilon) : \lim_{\epsilon \to 0} d_T(u_\epsilon) = D_T(\tilde{\mathbf{x}})
\]
\[
\Rightarrow d_T(u_\epsilon) = \frac{1}{f'(\epsilon)} \frac{p-1}{p} 3\pi^2 k \int_{\partial B_\epsilon} \left( \frac{\partial u_\epsilon}{\partial n} \right)^p.
\]
From these elements, we will show a numerical procedure that allow us to compute the limit $\epsilon \to 0$ in eq. (20) without performing an asymptotic analysis of the solution $u_\epsilon$ in relation to the parameter $\epsilon$. 

![Figure 1: model used to study the asymptotic behaviour with $\epsilon \to 0$.](image-url)
Through analysis of Fig. (2) we may observe straight lines which cross the origin of the graphic for every $p$ value. This fact allow us to infer that the integrand of eq. (22) behaves like a constant in relation to $\epsilon$. Therefore, function $f(\epsilon)$ may be chosen as $f(\epsilon) = -\pi \epsilon^2$.

![Figure 2: asymptotic behaviour of function $d_T(u_\epsilon) f'(\epsilon)$ in relation to the parameter $\epsilon$.](image)

Considering this analysis in eq. (22), we have the final expression of the topological derivative besides a constant, that is to say

$$d_T(u_\epsilon) = -\frac{p-1}{p} \frac{3^p k}{|\nabla u_\epsilon|^p} \left( \frac{1}{|\partial B_\epsilon|} \int_{\partial B_\epsilon} |\nabla u_\epsilon|^p \right)$$

$$\Rightarrow \lim_{\epsilon \to 0} d_T(u_\epsilon) = -C \frac{p-1}{p} \frac{3^p k}{|\nabla u|^p} , \quad (23)$$

Furthermore, the behavior of function $d_T(u_\epsilon)$ in relation to parameter $\epsilon$ is shown in Fig. (3), where we observe that the quotient

$$\frac{1}{|\partial B_\epsilon|} \int_{\partial B_\epsilon} |\nabla u_\epsilon|^p$$

$$\frac{|\nabla u|^p}{|\nabla u|^p}$$

(24)
tends to approximately 2 when $\epsilon$ diminishes.
Therefore, we can state the following conjecture:

**Conjecture 3** From analysis of Figs. (2, 3) the topological derivative for every \( p \geq 2 \) reads

\[
D_T(x) = -2 \frac{p - 1}{p} 3^{\frac{p}{2}} k |\nabla u|^p .
\]  

(25)

### 3.2.3 Numerical results

In the following example, we have a regular bar submitted to a rigid twist \( b = 1 \). The material properties of the bar are \( k = 1 \) and \( p = \{2, 3, 4, 6\} \). The domain \( \Omega \) is shown in Fig. (4a), where \( L = 50, a = 10 \) and \( R = 5 \). This problem is discretized using the quadratic *simplex* finite element, as can be seen in Fig. (4b).

In Fig. (5) are shown the absolute values of the topological derivative for \( p = \{2, 3, 4, 6\} \). From the analysis of this figure, we can propose a cross-section with a void as a solution for the problem. In fact, the holes shall be positioned where the topological derivative assumes the smallest values (white region in figures).
Figure 4: model and mesh with 12066 finite elements.

Figure 5: topological derivative.
3.2.4 Additional comments

In this section, the Topological-Shape Sensitivity Method was applied to compute the topological derivative for torsion of steady-state creep shafts (p-Poisson equation) considering the complementary dissipation energy as the cost function. From this approach, we have the final expression of the topological derivative as an integral on the boundary of the hole when its radius goes to zero. Finally, we have proposed a numerical procedure to compute the limit $\epsilon \to 0$. As a result, we have a methodology to compute the topological derivative for non-linear problems in general. For more details, see [18].

3.3 Inverse Poisson’s conductivity problem

The same theory developed to calculate the topological derivative via shape sensitivity analysis can be used to compute the sensitivity of a given cost function when a small incrustation, instead a hole, is introduced at a point in the domain. This sensitivity leads to a scalar function called configurational derivative.

Let us consider the domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) which is perturbed by introducing a small incrustation at an arbitrary point $\hat{\mathbf{x}} \in \Omega$. Therefore, we have a new domain $\tilde{\Omega}_\epsilon = \tilde{\Omega}_\epsilon \cup B_\epsilon$, where $B_\epsilon = B_\epsilon \cup \partial B_\epsilon$ is a ball of radius $\epsilon$ centered at the point $\hat{\mathbf{x}} \in \Omega$, $\Omega_\epsilon = \Omega - B_\epsilon$ and $\partial \Omega_\epsilon = \partial \Omega \cup \partial B_\epsilon$. In another words, $\Omega_\epsilon$ denotes the bulk material, $B_\epsilon$ the incrustation and their union represents the domain under consideration $\tilde{\Omega}_\epsilon$. Thus, we have the original domain without incrustation $\Omega$ and the new one $\tilde{\Omega}_\epsilon = \Omega_\epsilon \cup B_\epsilon$ with a small incrustation $B_\epsilon$. Considering a cost function $\psi$, the configurational derivative is defined as

$$D_C (\hat{\mathbf{x}}) := \lim_{\epsilon \to 0} \frac{\psi (\tilde{\Omega}_\epsilon) - \psi (\Omega)}{f (\epsilon)} ,$$

where $f (\epsilon)$ is a negative function that decreases monotonically so that $f (\epsilon) \to 0$ with $\epsilon \to 0^+$. Observe that the result of Theorem 1 can easily extend to compute the derivative given by eq. (26) via shape sensitivity analysis.

Now, the configurational derivative is computed in steady-state heat conduction problem on rigid solids, taking as the cost function a quadratic form of the difference between a measured (observed) and a calculated temperature. Let us consider a rigid solid represented by $\tilde{\Omega}_\epsilon = \Omega_\epsilon \cup B_\epsilon \subset \mathbb{R}^2$, with a small incrustation $B_\epsilon$ centered at $\hat{\mathbf{x}} \in \tilde{\Omega}_\epsilon$, submitted to an excitation $b$ in the domain $\tilde{\Omega}_\epsilon$. Considering continuity of the solution $u_\epsilon$ on $\partial B_\epsilon$, the variational formulation of the problem can be written as follows: find $u_\epsilon \in \mathcal{U}_\epsilon$, such that

$$\int_{\tilde{\Omega}_\epsilon} k^\epsilon \nabla u_\epsilon \cdot \nabla \eta + \int_{B_\epsilon} k^\epsilon \nabla u_\epsilon \cdot \nabla \eta = \int_{\tilde{\Omega}_\epsilon} b \eta - \int_{\Gamma_N} \tilde{q} \eta \quad \forall \eta \in \mathcal{V}_\epsilon ,$$

where $\mathcal{U}_\epsilon$ and $\mathcal{V}_\epsilon$ are defined by

$$\mathcal{U}_\epsilon = \{ u_\epsilon \in H^1 (\tilde{\Omega}_\epsilon) : u_\epsilon |_{\Gamma_D} = \overline{u} \}$$

and $\mathcal{V}_\epsilon = \{ \eta \in H^1 (\tilde{\Omega}_\epsilon) : \eta |_{\Gamma_D} = 0 \}$.
In addition, $\Gamma_D$ and $\Gamma_N$ are the Dirichlet and Neumann boundaries such that $\partial \Omega = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$; $\bar{u}$ and $\bar{q}$ are the temperature and heat flux prescribed on $\Gamma_D$ and $\Gamma_N$; $k^e$ and $k^i$ respectively are the thermal conductivity coefficients of bulk material (represented by $\Omega_s$) and incrustation (represented by $B_s$).

3.3.1 Shape sensitivity analysis

The cost function can be defined, already in the configuration $\tilde{\Omega}_\tau$, as

$$ J(u_\tau) := \psi(\tilde{u}_\tau) = \int_{\Gamma_{u^*}} (u_\tau - u^*)^2 , $$

(28)

where $u^*$ is a temperature measurement and $\Gamma_{u^*}$ is a line in $\Omega$ where the measurement of temperature is performed.

Considering the Lagrangian method and taking into account the Reynolds’ transport theorem, the derivative of the cost function, at $\tau = 0$, becomes

$$ \frac{d}{d\tau} J(u_\tau) \bigg|_{\tau=0} = \int_{\Omega_s} \Sigma_\varepsilon \cdot \nabla \mathbf{v} . $$

(29)

In this case, the Eshelby’s energy-momentum tensor $\Sigma_\varepsilon$ can be decomposed as

$$ \Sigma_\varepsilon|_{\Omega_s} := \Sigma^e_\varepsilon \quad \text{and} \quad \Sigma_\varepsilon|_{B_s} := \Sigma^i_\varepsilon . $$

(30)

where $\Sigma^e_\varepsilon$ and $\Sigma^i_\varepsilon$ are respectively given by

$$ \Sigma^e_\varepsilon = (k^e \nabla u_e \cdot \nabla \lambda_e - b \lambda_e) \mathbf{I} - k^e \left( \nabla u_e \otimes \nabla \lambda_e + \nabla \lambda_e \otimes \nabla u_e \right) , $$

(31)

$$ \Sigma^i_\varepsilon = (k^i \nabla u_i \cdot \nabla \lambda_e - b \lambda_e) \mathbf{I} - k^i \left( \nabla u_i \otimes \nabla \lambda_e + \nabla \lambda_e \otimes \nabla u_i \right) . $$

(32)

3.3.2 Configurational derivative calculation

It is well known that shape derivative only depends on the value of $\mathbf{v}$ at the boundary. In fact, it is straightforward to verify that the Eshelby tensor given by eqs. (31, 32) has null-divergence. Therefore, considering the definition of the velocity field given by eq. (3) and Theorem 1 (eq. 2), the configurational derivative yields

$$ D_C (\mathbf{x}) = -\lim_{\epsilon \to 0} \frac{k^e - k^i}{f'(\epsilon)} \int_{\partial B_s} \left( \frac{\partial u^\epsilon}{\partial t} \frac{\partial \lambda^\epsilon}{\partial t} + \frac{k^i}{k^e + k^i} \frac{\partial u^\epsilon}{\partial n} \frac{\partial \lambda^\epsilon}{\partial n} \right) , $$

(33)

where this last result (eq. 33) was obtained taking into account the continuity condition of the solutions $u_\epsilon$ ($u^\epsilon = u^i$) and $\lambda_\epsilon$ ($\lambda^\epsilon = \lambda^i$) on the boundary of the incrustation $\partial B_s$.

From an asymptotic analysis of the solutions $u_\epsilon$ and $\lambda_\epsilon$ with respect to the parameter $\epsilon$ [17, 13], it is possible to compute the limit $\epsilon \to 0$ in eq. (33) obtaining

$$ D_C (\mathbf{x}) = 2k^e \left( \frac{k^e - k^i}{k^e + k^i} \right) \nabla u \cdot \nabla \lambda , $$

(34)

where $f(\epsilon) = -\pi \epsilon^2$ and $u$ and $\lambda$ are the solutions of the state and adjoint equations, respectively, both defined in the original domain $\Omega$ (without incrustation).
3.3.3 Configurational derivative for several measurements

If we have several temperature measurements, the cost function can be defined as a sum of quadratics forms of the difference between observed and calculated temperatures for each measurement, that is

$$J(u) := \sum_{n=1}^{M} \int_{\Gamma_{u^n}} (u_n - u_n^*)^2,$$

(35)

being $u_n^*$ the $n$-th temperature measurement and $M$ the number of measurements. In this case, the configurational derivative becomes

$$D_C(\hat{x}) = 2k^e \left( \frac{k^e - k^i}{k^e + k^i} \right) \sum_{n=1}^{M} \nabla u_n \cdot \nabla \lambda_n,$$

(36)

where $u_n$ and $\lambda_n$ respectively are the solutions of state and adjoint equations defined in the original domain $\Omega$ for the $n$-th measurement.

3.3.4 Numerical results

The solutions $u$ and $\lambda$, defined in the original domain $\Omega$ (without incrustation), are computed through the Finite Element Method. Normally, measurement $u^*$ is obtained from experiments in laboratory. However, on the following examples, $u^*$ is also calculated through the Finite Element Method. Therefore, function $u^*$ represents the solution of the state equation defined on the domain including the incrustations that shall be identified. It should be mention that thermal conductivity coefficient of the incrustation $k^i$ is unknown. On the other hand, we assume that thermal conductivity coefficient of the bulk material $k^e$ is known. From analysis of eq. (26), the unknown incrustations that we want to identify must be positioned where the cost function is more sensible, that is, where $D_C(\hat{x})$ attains large absolute values.

Three examples is shown. In all of them, we consider a disk of radius equal to 0.5 with a prescribed temperature $\bar{u} = 0$ on its boundary $\Gamma_D$ with $k^e = 1.0$. In addition, the fixed region $\Gamma_{u^*}$ where the temperature measurement are observed is defined as

$$\Gamma_{u^*} = \{ x \in \Omega : |x - x_0| = 0.4 \},$$

(37)

where $x_0$ is the coordinate of the center of the disk. The conductivity coefficient of the incrustations is $k^i = 0.8$. Finally, considering a cartesian coordinate system $(x, y)$, the excitation $b$ is defined like a wave in the following manner:

$$\begin{cases} 
  b := \sin[(x \cos \theta + y \sin \theta) n \pi] & \text{for} \quad 0^\circ \leq \theta < 180^\circ \\
  b := \cos[(x \cos \theta + y \sin \theta) n \pi] & \text{for} \quad 180^\circ \leq \theta < 360^\circ 
\end{cases},$$

(38)
where \( n \) is the parameter associated to the frequency of the wave and \( 0 \leq \theta < 2\pi \) is the angle of incidence of the wave. Also, we take 120 temperature observations \((M = 120\) in eq. 36) and we assume an uniform distribution of the measurements on \( \Gamma_{u^*} \) with step \( \Delta \theta = 2\pi/M = \pi/60 \).

**Example 1:** In this case, the target domain has a circular incrustation centered on \( x_0 \) as shown in Fig. (6a). Taking \( n = 30 \), we obtain the sum of fields \( \nabla u_m \cdot \nabla \lambda_m \) depicted in Fig. (6b).

![Figure 6: example 1 - one circular incrustation.](image)

**Example 2:** The target domain for this case has an L-shaped incrustation as may be seen in Fig. (7a). The sum of fields \( \nabla u_m \cdot \nabla \lambda_m \) considering \( n = 30 \) is shown in Fig. (7b).

![Figure 7: example 2 - one L-shaped incrustation.](image)
Example 3: In this last case, the target domain has three circular incrustations as shown in Fig. (8a). The sum of fields $\nabla u_m \cdot \nabla \lambda_m$ considering $n = 30$ is shown in Fig. (8b).

![Diagram of example 3](image)

Figure 8: example 3 - three circular incrustations ($n = 30$).

In Fig. (9), we shown the sum of fields $\nabla u_m \cdot \nabla \lambda_m$ obtained for a frequency $n = 60$.

![Diagram of example 3](image)

Figure 9: example 3 - sum of fields $\nabla u_m \cdot \nabla \lambda_m$ obtained considering $n = 60$ for the case with three circular incrustations.

From analysis of Figs. (6, 7, 8, 9) we can observe that the configurational derivative is able to clearly identify the shape and the topology of an unknown incrustation field.
3.3.5 Additional comments

The main goal of this section is to show the applicability of the configurational derivative (eq. 26) in problems where the shape and topology of incrustations must be characterized. This new approach can be used to devise an alternative reconstruction method: since the solution of the inverse problem is given by the domain \( \tilde{\Omega} \), which minimizes the functional \( \psi(\tilde{\Omega}) \), we can choose the points in \( \Omega \) (which does not contain any incrustation) where the value of the configurational derivative attains its highest absolute values. Note that these points give the largest decrease in the cost functional when incrustations are placed on them. In particular, considering the steady-state heat conduction problem, we have shown that the configurational derivative allow us to identify the shape and topology of the unknown incrustations. On the other hand, classical approach for this kind of problem based on shape optimization has an intrinsic limitation: the number of incrustation in the bulk material must be known \textit{a priori}. In order to overcome this difficulty the configurational derivative evolves as an interesting alternative. In fact, the unknown field is identified even when there are several incrustations. This issue is also discussed in [7]. Finally, we would like to say that the configurational derivative can also be utilized as a new approach for mechanical modelling of problems such that cavitation, plasticity, ductile fracture, phase change, damage among others. For more details see [20].

4 CONCLUSIONS

In this work, we have used the Topological-Shape Sensitivity Method as a systematic methodology for computing the topological derivative for several engineering problems. More specifically, we have proposed a numerical procedure to compute the topological derivative in non-linear problems. In addition, we have introduced the configurational derivative that gives the sensitivity of a cost function when an incrustation is introduced at a point of the domain. This idea leads to a new reconstruction method to solve a class of inverse problems. Therefore, we have verified that the topological derivative concept in a promising and challenging research area with potential application in topology optimization, inverse problem and mechanical modeling.

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