PRECONDITIONED SCHEMES FOR NONSYMMETRIC
SADDLE-POINT PROBLEMS

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Abstract. In this paper, we present an effective preconditioning technique for solving nonsymmetric saddle-point problems. In particular, we consider those saddle-point problems that arise from the numerical solution of the mixed finite element discretization of particulate flows – flow of solid particles in incompressible fluids. These indefinite linear systems are solved using a preconditioned Krylov subspace method with an indefinite preconditioner. This creates an inner-outer iteration, in which the inner iteration is handled via a preconditioned Richardson scheme. We provide an analysis of our approach that relates the convergence properties of the inner to the outer iterations. Also “optimal” approaches are proposed for the construction of the Richardson’s iteration preconditioner. The analysis is validated by numerical experiments that demonstrate the robustness of our scheme, its lack of sensitivity to changes in the fluid-particles system, and its “scalability”.

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SUMMARY

Many scientific applications require the solution of saddle point problems of the form

\[
\begin{bmatrix}
A & B \\
B^T & 0
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix} =
\begin{bmatrix}
a \\
b
\end{bmatrix},
\]

(1)

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) with \( m \leq n \), and where the \((n + m) \times (n + m)\) coefficient matrix

\[
A = \begin{bmatrix}
A & B \\
B^T & 0
\end{bmatrix},
\]

(2)

is assumed to be nonsingular.

Such systems are typically obtained when “Lagrange multipliers” or mixed finite element discretization techniques are employed. Examples of these include, but are not limited to, the equality-constrained quadratic programming problems, the discrete equations which result from the approximation of elasticity problems, Stokes equations, and the linearization of Navier-Stokes equations. When the matrix \( A \) is symmetric and positive definite, the problem (1) has \( n \) positive and \( m \) negative eigenvalues, with well defined bounds. If the matrix \( A \) is symmetric indefinite or nonsymmetric, little can be said about the spectrum of the indefinite matrix \( A \).

Much attention has been paid to the case when \( A \) is symmetric positive definite, and more recently to the case when \( A \) is nonsymmetric. In this paper, \( A \) is assumed to be nonsymmetric and \( B \) is of full column rank. Here, we adopt one of the symmetric indefinite preconditioners proposed by Golub and Wathen for solving (1) directly via a preconditioned Krylov subspace method, such as GMRES, given by

\[
\mathcal{M} = \frac{1}{2}(A + A^T) = \begin{bmatrix}
A_s & B \\
B^T & 0
\end{bmatrix}.
\]

(3)

Here, \( A_s \) is the symmetric part of \( A \), i.e., \( A_s = (A + A^T)/2 \). The motivating application in our paper produces an \( A_s \) which has the following properties: (a) positive definite and irreducibly diagonally dominant, i.e., \( A_s = [a^{(s)}_{ij}] \) is irreducible, and \( a^{(s)}_{ii} \leq \sum_{j \neq i} |a^{(s)}_{ij}| \) with strict inequality holding for at least one \( i \), and (b) \( \| A_s \|_F \geq \| A_{ss} \|_F \) where \( \| \cdot \|_F \) denotes the Frobenius norm, where \( A_{ss} = (A - A^T)/2 \) is the skew symmetric part of \( A \). Thus, the preconditioner \( \mathcal{M} \) is nonsingular, and the Schur complement, \(-(B^T A_s^{-1} B)\), is symmetric negative definite.
The application of the preconditioner $\mathcal{M}$ in each Krylov iteration requires the solution of a linear system of the form

$$
\begin{bmatrix}
A_s & B \\
B^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
f \\
g
\end{bmatrix}.
$$

(4)

The focus of our study is the development of a preconditioned Richardson iterative scheme for solving the above symmetric indefinite system (4) in a nested iterations setting that assures the convergence of the inner iterations.

This system can be reformulated as

$$
A_s x = f - B y,
$$

(5)

$$
(B^T A_s^{-1} B)y = B^T A_s^{-1} f - g.
$$

(6)

Thus, one may first solve (6) to obtain $y$, (inner iteration), then solve (5) to get $x$.

Using a conjugate gradient algorithm for solving (6), and (5), we create an inner-outer iterative scheme. This is the approach used in the classical Uzawa scheme. It turns out that in order to ensure convergence of the outer iteration, it is necessary to solve systems in the inner iteration with relatively high accuracy.

For large-scale applications, such as the numerical simulation of particulate flows, solving linear systems involving $A_s$ or $(B^T A_s^{-1} B)$ is not practical, as the action of $A_s^{-1}$ must be computed on various vectors. Consequently, the approach we adopt here is to replace the cost of computing the action of $A_s^{-1}$ by the cost of evaluating the action of some other “more economical” symmetric positive definite operator $\hat{A}^{-1}$ which approximates $A_s^{-1}$ in some sense.

Thus, the linear system (5) is solved via the iteration

$$
x_{k+1} = (I - \hat{A}^{-1} A_s)x_k + \hat{A}^{-1} \tilde{f},
$$

(7)

where $\tilde{f} = f - B y$ and $\hat{A}$ is an appropriate symmetric positive definite splitting that assures convergence, i.e., $\alpha = \rho(I - \hat{A}^{-1} A_s) < 1$, where $\rho(\cdot)$ is the spectral radius.

Similarly, we replace $A_s$ by $\hat{A}$ in (6) and solve the resulting “inexact” system,

$$
(B^T \hat{A}^{-1} B)y = B^T \hat{A}^{-1} f - g.
$$

(8)

instead of the original system (6), via the iteration

$$
y_{k+1} = [I - \hat{G}^{-1} (B^T \hat{A}^{-1} B)]y_k + \hat{G}^{-1} \tilde{s},
$$

(9)
where \( \mathbf{s} = B^T \hat{A}^{-1} \mathbf{f} - \mathbf{g} \), and \( \hat{G}^{-1} \) is an inexpensive symmetric positive definite approximation of the inverse of the inexact Schur complement \( (B^T \hat{A}^{-1} B)^{-1} \) that assures convergence of (9), i.e., \( \beta = \rho(I - \hat{G}^{-1}(B^T \hat{A}^{-1} B)) < 1 \). Moreover, \( \hat{G}^{-1} \) is chosen such that \( (I - \hat{G}^{-\frac{1}{2}}(B^T \hat{A}^{-1} B) \hat{G}^{-\frac{1}{2}}) \) is positive definite.

Similarly, if we define the symmetric preconditioner \( \hat{\mathcal{M}} \) to the system (4) as

\[
\hat{\mathcal{M}} = \begin{bmatrix}
\hat{A} & B \\
B^T & -\hat{G} + (B^T \hat{A}^{-1} B)
\end{bmatrix},
\]

(10)
we obtain the following preconditioned Richardson iterative scheme for solving (4)

\[
\begin{bmatrix}
x_{k+1} \\
y_{k+1}
\end{bmatrix} = \begin{bmatrix}
x_k \\
y_k
\end{bmatrix} + \begin{bmatrix}
\hat{A} & B \\
B^T & -\hat{G} + B^T \hat{A}^{-1} B
\end{bmatrix}^{-1} \begin{bmatrix}
\mathbf{f} \\
\mathbf{g}
\end{bmatrix} - \begin{bmatrix}
A_s & B \\
B^T & 0
\end{bmatrix} \begin{bmatrix}
x_k \\
y_k
\end{bmatrix},
\]

(11)
that is convergent if and only if \( \rho(I - \hat{\mathcal{M}}^{-1}\mathcal{M}) < 1 \).

Thus, our proposed nested iterative scheme is shown in Figure 1 in which the outermost iteration is that of a Krylov subspace method (we use restarted GMRES throughout this paper), and the preconditioning operation itself is doubly nested. Our focus here is the development of an algorithm for the most inner iteration, i.e. solving systems involving the Golub-Wathen (GW) preconditioner using the preconditioned Richardson iteration (11).

In the Golub-Wathen study, systems of the form \( \mathcal{M} \mathbf{z} = \mathbf{r} \), in the inner-most loop \( \{c\} \) of Figure 1, are solved using a direct scheme. In our study, the monotone convergence of our inner iteration (11) is guaranteed, and the performance of our nested scheme in Figure 1 does not degrade as the mesh size decreases.

Moreover, the construction of the preconditioner \( \mathcal{M} \) of the Richardson iteration is simple and economical.

In this paper, we analyze the iterative scheme (11) and show that a sufficient condition for monotone convergence is \( \max\{\alpha, \beta\} < (\sqrt{5} - 1)/2 \), and thus relating the rate of convergence of the inner iterations to the outer iteration, even though (9) is not the iteration that corresponds to the exact system (6) to be solved but to a modified one (8), which, we will show, is not required to be solved accurately.

We use a simple explicit approximate inverse \( A_0^{-1} \) of \( A_s^{-1} \) for which \( \alpha_0 = \rho(I - A_0^{-1} A_s) < 1 \) and obtain an iteration for improving the convergence rate of (7). The matrix \( \hat{G}^{-1} \) is not formed explicitly and the solution of systems involving \( \hat{G} \) is achieved via the CG scheme, thus the only operations involved in the proposed nested iterative scheme (11) are matrix-vector multiplications and vector operations.
Our preconditioning strategy of the inner Richardson iteration is motivated by the study of Bank, Welfert and Yserentant on a class of iterative methods for solving saddle-point problems. We extend it in this paper with some new results and a new analysis that relates the proposed iterative scheme to Uzawa’s method. Further, we use our scheme for solving those *indefinite* linear systems that arise from the mixed finite element discretization of 2D particulate flow problems, using P2-P1 type elements.

In this paper, we introduce our motivating application – that of the direct numerical simulation of particulate flows, present our proposed nested iterative scheme, and analyze its convergence properties. We propose “optimal” approaches for the construction of $\hat{A}^{-1}$ and $\hat{G}^{-1}$ approximating $A_s^{-1}$ and $(B^T \hat{A}^{-1} B)^{-1}$, or their actions on vectors, so as to assure convergence of our scheme. We also demonstrate the robustness of our nested iterative scheme as a preconditioner, its lack of sensitivity to changes in the fluid-particles systems, and its “scalability”.

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**Figure 1. A Nested Iterative Scheme.**

(a) Solve
\[
\begin{bmatrix}
A & B \\
B^T & 0
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix} =
\begin{bmatrix}
a \\
b
\end{bmatrix} : A \neq A^T
\]

via a Krylov subspace method.

(b) Preconditioner: $\mathcal{M} = \begin{bmatrix}
A_s & B \\
B^T & 0
\end{bmatrix}$

$A_s = (A + A^T)/2$

(c) Solve $\mathcal{M}z = r$

Use the preconditioned Richardson iteration

$z_{k+1} = z_k + \hat{\mathcal{M}}^{-1}(r - \mathcal{M}z_k)$

where $\hat{\mathcal{M}} = \begin{bmatrix}
\hat{A} & B \\
B^T - \hat{G} + (B^T \hat{A}^{-1} B)
\end{bmatrix}$